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ABSTRACT

Research reports from the annual conference of the International Group for the Psychology of Mathematics Education include: "La construcion algorithmique: niveaux ou stades?" (Mesquita); "For Establishing the Generality of Conjectures" (Miyazaki); "Teachers' Attitudes Towards Mathematics and Mathematics Teaching: Perspectives across Two Countries" (Moreira); "World Problems--the Construction of Multiplicative Structures" (Morgado); "Reconstruction of Mathematics Education: Teachers' Perceptions of and Attitudes to Change" (Mousley); "The Falsifiability Criterion and Refutation by Mathematical Induction" (Movshovitz-Hadar); "Young Children's Division Strategies" (Murray; Olivier; Human); "'It Makes Sense if You Think about How the Graphs Work. But in Reality...'" (Nemirovsky; Rubin); "Two-Step Problems--Research Findings" (Nesher; Hershkovitz); "Early Conceptions of Division. A Phenomenographic Approach" (Neuman); "Can Epistemological Pluralism make Mathematics Education More Inclusive?" (Neville); "The Pupil as Teacher: Analysis of Peer Discussions, in Mathematics Classes, between 12-Year-old Pupils and the Effects of Their Learning" (Newman; Pirie); "Teacher Attitudes and Interactions in Computational Environments" (Noss; Hoyles); "Children's Understanding of Measurement" (Nunes; Light; Mason); "A 3-Dimension Conceptual Space of Transformations for the Study of Intuition of Infinity in Plane Geometry" (Nunez); "The Status of Children's Construction of Relationships" (O'Brien); "Intra-Individual Differences in Fractions Arithmetic" (Ohlsson; Bee); "Construction of Procedures for Solving Multiplicative Problems" (Orozco; Hormaza); "Transfer in Learning 3D Reference System: From Interaction "Pupils as Expert System Developers" (Osta); "Levels of Knowledge about Signed Numbers: Effects of Age and Ability" (Peled); "Representations du probleme de mathematiques chez des enfants de 7 a 10 ans" (Perrin-Glorian); "Teaching as Meta-Communication" (Pimm); "Folding Back: Dynamics in the Growth of Mathematical Understanding" (Pire; Kieren); "Enseigner les mathematiques en premiere annee secondaire apres l'evaluation nationale francaise" (Pluvinage; Rauscher; Dupuis); "Etude

des modeles implicites mis en ouvre par les enfants lors de la resolution de problemes complexes mettant en jeu une reconstruction d'une transformation arithmetique" (Poirier; Bednarz); "Classroom Aspects which Influence Use of Visual Imagery in High School Mathematics" (Presmeg); "Computer Activities in Mathematical Problem Solving with 11-14 Years Old Students: the Conditional Structure Learning" (Reggiani); "Symbolising and Solving Algebra Word Problems: the Potential of a Spreadsheet Environment" (Rojano; Sutherland); "Damien: a Case Study of a Reorganization of His Number Sequence to Generate Fractional Schemes" (Saenz-Ludlow); "The Use of Language in the Context of School Mathematics" (Sakonidis; Bliss); "Emergent Goals in Everyday Practices: Studies in Children's Mathematics" (Saxe); "Teachers' and Students' Beliefs and Opinions about the Teaching and Learning of Mathematics in Grade 4 in British Columbia" (Schroeder); "Problem Solving and Thinking: Constructivist Research" (Schultz); "Assessment of Thoughts Processes with Mathematical Software" (Schwarz; Dreyfus); "Spontaneous Strategies for Visually Presented Linear Programming Problems" (Shama; Dreyfus); "Initial Development of Prospective Elementary Teachers' Conceptions of Mathematics Pedagogy" (Simon); "The Effects on Students' Problem Solving Behaviour of Long-term Teaching through a Problem Solving Approach" (Stacey); "The Relationship Between Mental Models in Mathematics and Science" (Stavy; Tirosh); "Pupils as Expert System Developers" (Stevenson; Noss); "Drawing--Computermodel--Figure Case Studies in Students' Use of Geometry-Software" (Strasser; Capponi); "Overcoming Overgeneralizations: the Case of Communitativity and Associativity" (Tirosh; Hadass; Movshovitz-Hadar); "First Steps in Generalization Processes in Algebra" (Ursini); "Translation Processes Solving Applied Linear Problems" (Van Streun); "Graphical Environment for the Construction of Function Concepts" (Wenzelburger); "The Potential for Mathematical Activity in Tiling: Constructing Abstract Units" (Wheatley; Reynolds); "The Equal Sign Goes Both Ways. How Mathematics Instruction Leads to the Development of a Common Misconception" (Wolters); "Learning in an Inquiry Mathematics Classroom" (Wood); "The Role of Peer Questioning During Class Discussion in Second Grade Mathematics" (Yackel); "The Effect of Graphic Representation: An Experiment Involving Algebraic Transformations" (Yerushalmy; Gafni); and "In What Ways Are Similar Figures Similar?" (Zaslavsky).

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VOLUME III

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CONTENTS OF VOLUME III

LA CONSTRUCTION ALGORITHMIQUE : NIVEAUX OU STADES ? *

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La construction algorithmique a-t-elle, comme d'autres notions géométriques, une genèse repérable par des stades d'acquisition de type piagetien, associés à l'état de développement de chaque individu, ou, au contraire, est-elle liée à la représentation que les individus se font eux-mêmes de ce qu'est un algorithme? L'étude que nous avons conduite, auprès d'environ trois cents élèves de 12 à 15 ans, semble privilégier la deuxième hypothèse ; trois types d'appréhension y sont identifiés et décrits.

This paper concerns the question of the genesis of algoritmic tasks in geometry. Our study, conducted with about three hundred 12 to 15 years-old pupils, suggests that this genesis is associated with the pupils' algorithm representation, rather than the development of individuals : three levels of apprehension of an algorithm are identified and described.

Une des principales difficultés suscitées par une construction géométrique, et notamment par une construction algorithmique, est due au fait suivant : l'algorithme est une procédure de construction lié à des objects mathématiques, abstraits et généraux, mais sa concrétisation ne peut être exprimée que par une configuration spécifique, mettant en jeu des objets concrets et particuliers. Cette concrétisation exige que l'on passe au régistre figuratif, où chaque concept est traduit visuellement par des objets particuliers.

La généralité de la procédure, et donc de l'algorithme, n'est pas directement appréhendée sur sa concrétisation. Le registre figuratif -utilisé nécessairement pour effectuer la construction- n'est pas le plus approprié pour saisir ce caractère général de l'algorithme. Cet écart entre la particularité inhérente au registre figuratif et la généralité de la procédure peut expliquer des difficultés conceptuelles dans l'acquisition de l'algorithme . un obstacle conceptuel essentiel peut alors résulter de l'utilisation du registre figuratif (R. Duval, 1988). En effet, les élèves peuvent associer à l'algorithme des contraintes qui relèvent seulement de sa concrétisation. En d'autres termes, les élèves peuvent avoir une appréhension d'un algorithme trop liée à la configuration qui sert à exemplifier celui-ci. Des différents types d'appréhension d'un algorithme peuvent alors exister chez les élèves.

*Notre première hypothèse de travail est que la diversité des appréhensions que les élèves ont de l'algorithme s'explique par l'écart entre **figure** et **procédure***

*Recherche développée avec le support de Fundação Gulbenkian, Lisbonne

Pour saisir cette diversité d'appréhensions, nous avons fait varier les conditions d'application de l'algorithme. Il nous a semblé qu'en demandant une application de l'algorithme dans des cas où la distinction entre les particularités de la figure et la généralité de la procédure était exigée - ce qui nous avons appelés les cas-limite- était suffisant pour faire apparaître ces différentes appréhensions.

Notre deuxième hypothèse est que l'utilisation de l'algorithme dans des cas-limite révèle la compréhension réelle de l'algorithme.

Pour cela, nous avons choisi une situation géométrique, où une tâche algorithmique[1] était demandée à des élèves de collège. Dans la question retenue, -inspirée en F. Pluvinage et al. (1985)- une procédure permettant d'inscrire un carré dans un triangle était donnée à l' aide d'"un film". Un questionnaire a été construit , avec les tâches suivantes

- tâche de reproduction : inscription d'un carré dans un triangle, dans un cas de figure proche de celui utilisé dans le film ; la description de la reproduction était également demandée ;
- tâche d'adaptation, concernant l'inscription d'un carré dans deux cas-limite :
  - -i) l'inscription d'un carré dans un triangle rectangle ;
  - -ii) l'inscription d'un carré dans un triangle avec un angle obtus ;

autre que les réalisations des constructions, nous avons demandé aux élèves d'argumenter sur la possibilité de généralisation de la procédure.

Analyse a priori

1. Mathématiquement, la situation retenue pour notre étude est une application du concept d'homothétie. Le but est la construction d'un carré inscrit à un triangle. Le problème de cette inscription en soi-même est assez complexe, même pour les futurs enseignants (G. Glaeser, 1971). En effet, l'homothétie de centre A, sommet du triangle, appliquée à un carré auxiliaire (donc un côté, BC, est commun au triangle) permet l'inscription d'un carré (homothétique au carré auxiliaire) dans le triangle ABC, donné au départ (cf. fig.1).

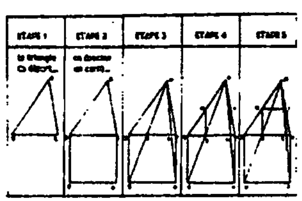


Figure 1

[1]Nous désignons par tâche algorithmique un ensemble de tâches, où une succession de règles de construction y est donnée.

Sous réserve, bien entendu, de convexité de la région ACDEBA .

2. La tâche d'adaptation entraîne des obstacles, dont nous parlerons par la suite. En effet, le choix du côté sur lequel on effectue la construction peut entraîner des difficultés de nature heuristique, mais aussi conceptuel.

Cas du triangle rectangle. La construction sur un petit côté exige une adaptation de l'algorithme proposé. Dans ce cas, on peut faire la construction soit à partir d'un petit côté, soit à partir de l'hypoténuse.

Cas du triangle avec un angle obtus. Dans ce cas, seule la construction à partir du côté opposé à l'angle obtus permet l'inscription d'un carré dans le triangle. En fait, **l'application de l'algorithme exige la convexité** de la région formée par le triangle et par le carré auxiliaire. Sous cette condition, le carré auxiliaire a pour homothétique, de centre convenable, un carré inscrit dans le triangle.

La construction à partir du côté opposé à l'angle obtus se heurte à deux obstacles:

- un premier, d'ordre *heuristique*, peut être décrit comme l'*idée* d'utiliser un autre côté;
- le deuxième, d'ordre *perceptif*, est lié au fait que la configuration perceptive est dans ce cas assez différente de la configuration suggérée par le film.

Ces obstacles font surgir naturellment un problème : celle de la *légitimité* du changement de base. *En changeant de base, applique t-on encore le même algorithme?* - tel est le problème que certains élèves peuvent se poser. La résolution de ce problème dépend de l'appréhension que l'on a de l'algorithme.

Méthodologie

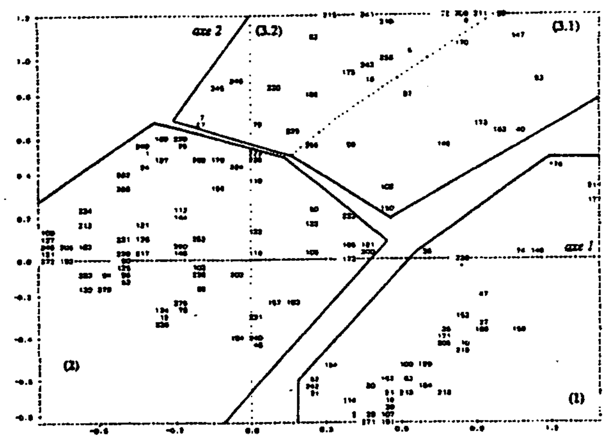
Le questionnaire a été passé à tous le élèves de cinquième, quatrième et troisième (élèves de 12 à 15 ans) d'un collège de la banlieue de Strasbourg, soit à 283 élèves. Il a été divisé en deux parties, chaque partie étant passée pendant une heure de cours, en deux séances suivies. Les questionnaires ont été "mis à l'épreuve" par des passation expérimentales, faites avec des élèves de divers niveaux scolaires (forts-faibles), de la tranche d'âge des élèves de la passation finale.

Pour l'analyse des résultats, nous avons utilisé des méthodes d'analyse factorielle de correspondances multiples et des méthodes de classification automatique, existant dans SPAD (Système Portable pour l'Analyse de Données), disponible au Centre de Calcul de Strasboug.

Analyse des résultats

Nous nous centrons ici dans les résultats didactiques de notre étude ; une analyse plus detaillée est décrite en A.Mesquita (1989).

Les élèves semblent se grouper en trois agrégats (cf. graphique 1). Un premier agrégat est formé par 90 élèves (32% de la population) échouant à la tâche d'adaptation et de généralisation, mais réussissant à la tâche de reproduction. Un deuxième agrégat regroupe les 127 élèves (45% de la population) en réussite à la tâche d'adaptation. Le troisième agrégat regroupe les 66 élèves (23% de la population) qui échouent aux tâches de reproduction et d'adaptation, par l'utilisation d'autres procédures, ou qui réussissent à la tâche de reproduction, tout en y donnant des descriptions inconsistantes (sous-agrégats 3.1 et 3.2, respectivement).



Légende: (1) : Agrégat 1 (2) : Agrégat 2 (3.1) : Sous-agrégat 3.1 (3.2) : Sous-agrégat 3.2

Graphique 1

Les trois types d'appréhension d'un algorithme

L'étude statistique des résultats et une analyse detaillée des réponses des élèves suggèrent l'existence de trois types d'appréhension d'un algorithme. A noter que ces types d'appréhension n'apparaissent pas liés à l'âge des élèves. A noter encore l'importance des présomptions[2] au sens de Grize (1982) dans les justifications de élèves, en liaison étroite avec le type d'appréhension, qui apparaissent ainsi comme différentes *représentations* de algorithme, plutôt que comme des stades (au sens piagetien). Ces différents types d'appréhension prennent en compte les performances atteintes et les types de désignations employées.

[2] C'est-à-dire, aux jugements préalables au discours, qui ne sont pas mis en cause par les élèves.

Une première appréhension est de type ***figural***. Les contraintes figurales sont appréhendées, d'une façon rigide et exclusive. A ce type d'appréhension correspond une vision physique de l'espace, un espace qui est perçu comme matériel et immobile. La performance la meilleure qui peut être obtenue dans ce cas est celle d'une reproduction d'une construction algorithmique. dans les cas d'une configuration perceptive proche de la situation donnée. La construction est faite dans ce cas comme s'il s'agissait d'un *dessin*. La désignation associée à ce type d'appréhension est une désignation spatiale nominale (du genre nom propre) liée à un emplacement absolu (une sorte de marquage, comme sur le terrain). Nous retrouvons ici le type de *repérage géographique* mentionné par C. Laborde (1982). La désignation n'est pas fonctionnelle, elle ne sert pas à établir des relations entre les objets, elle est plutôt une *nomination* de ceux-ci. Aussi la représentation de l'espace de travail associé est en conformité avec les autres caractéristiques : il est perçu comme un plan de dessin avec une existence physique, matérielle et fixe. Cette appréhension correspond au troisième agrégat de la classification.

Un deuxième type d'appréhension de l'algorithme est l'appréhension ***fonctionnelle***, intermédiaire. Dans ce type d'appréhension les contraintes -algorithmiques ou figurales- sont perçues, mais des confusions peuvent subsister : par exemple, une position relative (liée à une configuration perceptive) peut être perçue comme contrainte algorithmique. Liée à la distinction entre les deux types de contraintes, il y a la prise en compte du statut, c'est-à-dire, de la légitimité des adaptations nécessaires à une réalisation de la construction, qui apparaît comme un *schéma*, traduisant quelques relations. Ainsi, certaines constructions algorithmiques peuvent être réalisées (tout en maintenant la même configuration perceptive qui est dominante). Par exemple, dans la situation algorithmique que nous avons utilisée, l'adaptation était faite en annonçant qu' *"il fallait tourner la feuille"*. L'espace de travail est aussi un espace en transition. Il apparaît comme un espace physique, mais mobile. Le type de désignation utilisé ici permet un *repérage relatif*, lié à un emplacement relatif (et non plus absolu). Les désignations apparaissent comme des invariants par rapport à certaines transformations, par exemple, à la rotation. Ce deuxième type d'appréhension correspond au premier agrégat de la classification.

Une dernière appréhension est de type ***structural***. Dans ce cas, le jeu des contraintes est bien perçu et la distinction entre contraintes algorithmiques et figurales est claire ; elle permet la réalisation des tâches d'adaptation indépendamment de la configuration perceptive. L'espace de travail apparaît comme un espace projectif, mathématique, mobile, sans existence matérielle. La superposition de formes y est admise et l'obstacle du dédoublement, c'est-à-dire, de l'existence d'un côté commun au triangle et au carré, avec des fonctions différentes, en est franchi. La désignation des lettres est ici *relationnelle*, non liée à l'emplacement. Les lettres apparaissent comme des variables muettes, comme un code permettant d'exprimer des relations. C'est cette

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dernière appréhension qui peut être considérée comme *algorithmique*. Dans notre classification, ce type d'appréhension correspond au deuxième agrégat.

Le tableau suivant résume les caractéristiques de chaque type d'appréhension :

TABLEAU SYNOPTIQUE DES TYPES D'APPREHENSION

Appréhension	Caractéristiques	Performances	Désignation Emplacement
figurale *dessin*	° contraintes figurales perçues ° espace physique (matériel, fixe)	° reproduction (même configuration perceptive)	° géographique ° absolu
fonctionnelle *schéma*	° contraintes figurales et algorithmiques perçues, confusions possibles ° espace de transition (matériel, mobile)	° certaines adaptations (configurations perceptives semblables) ° invariance par rapport à la rotation	° fonctionnelle ° relatif
structurale *algorithme*	° contraintes figurales et algorithmiques bien demarquées ° espace mathématique (immatériel, mobile)	° adaptation (toutes configurations)	° relationnelle ° relatif

Remarques

1) *sur l'utilisation des notations*

On retrouve dans le type de désignation utilisé par les élèves une évolution semblable à l'historique : on distinguait un point des autres en parlant du "point (situé) en A" ce qui devenait "le point A" et enfin simplement "A" (H. Freudenthal, 1985, p. 147). "A" peut être interprété comme un nom propre, à savoir le nom du point de la figure en question auprès duquel se trouve la lettre A. Mais étant donné la généralité des énoncés géométriques où A peut être un point quelconque, les lettres devenaient des noms ambigus de points, donc des symboles de variables. Ainsi, la désignation *géographique* correspond à la phase du "point situé en A", comme la désignation *relationnelle* correspond à "A" tout court. La désignation *fonctionnelle* correspond à une étape intermédiaire entre A comme nom propre, et A comme nom "ambigu", c'est-à-dire, comme variable. C'est ce mode de désignation relationnelle qu'on retrouve en algèbre.

2) *sur l'imbrication entre appréhension de l'algorithme et le type de désignation utilisé*

Cette imbrication montre l'interaction, voire la complementarité, entre la conceptualisation et la symbolisation : la représentation d'un objet par un symbole entraîne l'abandon de certains

éléments, attachés à l'objet, et qui ne sont pas pertinents pour le problème. Dans le cas des situations algorithmiques, les éléments sur lesquels on travaille sont, par nature, des symboles. Pour cerner la nature d'un algorithme, une conceptualisation est alors nécessaire ; cette conceptualisation est liée à l'appréhension de contraintes (algorithmiques et non plus figurales) et au type d'objets géométriques sur lesquels on travaille et par conséquent à leur désignation.

Conclusion et discussion

Le double statut

Les résultats de la tâche algorithmique montrent la complexité d'une telle tâche. En fait, une tâche qui peut à première vue apparaître comme facile, soulève des difficultés diverses. En premier lieu, toute *tâche algorithmique* présente, par sa nature, une ambiguité fondamentale : elle apparaît à la fois comme la **conceptualisation d'une construction**, s'appuyant sur des objets géométriques généraux et abstraits, et comme la **concrétisation de cette même construction**, utilisant dans ce cas des objets géométriques particuliers et bien identifiés. Il faut cependant reconnaître que le registre figuratif par lui-même, renforce ce type d'ambiguité : tout seul, il ne permet pas de distinguer la procédure générale et la réalisation particulière.

Ce double statut entraîne des difficultés dans l'appréhension des contraintes mises en jeu dans un algorithme géométrique, c'est-à-dire, dans l'identification des règles opératoires propres à l'algorithme d'une part, et à réalisation sur une figure particulière, d'autre part. Nous les avons distinguées en les désignant respectivement comme contraintes algorithmiques et comme contraintes figurales, au sens de R. Duval (1988). Plus précisément, les difficultés résident dans la reconnaissance du type de contraintes en présence.

Liée à cette difficulté, il y a la question de la légitimité des adaptations nécessaires à la réalisation de la construction. Autrement dit, la question est de savoir jusqu'à quel point les modifications des contraintes figurales sont admises. A ce propos, nous rappelons le comportement fréquent des élèves sur la possibilité de prendre une base non horizontale du triangle.

Niveaux ou stades?

Trois types d'appréhension ont être observés, en liaison étroite avec la discrimination dés contraintes figurales et algorithmiques. Ces types d'appréhension sont liés aux représentations que les élèves se font eux-mêmes de ce qu'est un algorithme. L'indépendance par rapport à l'âge suggère que ces types d'appréhension apparaissent plutôt liés aux présomptions des individus (au sens de Grize), qu'à l'âge de ces derniers. A chaque type d'appréhension correspond un changement dans le mode de désignation employé, ce qui semble aussi confirmer l'idée que les différences d'appréhension d'un algorithme sont essentiellement

représentationnelles. Au contraire de certaines genèses de notions géométriques étudiées par Piaget et ses collaborateurs (1973,1981) on ne peut pas ainsi les interpréter comme des stades différents par lequel les individus devraient passer dans l'apprentissage des démarches algorithmiques. Ceci d'ailleurs concorde avec d'autres études. Les recherches de J. -M. Dolle et de son équipe (cités par Baldy R. et al., 1988) sur la genèse de la représentation graphique de la perspective "font apparaître qu'il n'y a pas une genèse homogène de la structuration de l'espace et que le repérage des propriétés de l'espace graphique en perspective [...] ne s'effectue pas séparément selon une genèse uniforme et repérable par des stades d'acquisition. Ces propriétés n'ont pas une existence indépendante les unes des autres ; elles n'existent qu'en composition". (ibid., p.151).

REFERENCES

BALDY R. et al. (1987) Activités cognitives dans l'apprentissage et l'utilisation du dessin technique - Etat des travaux, in G. Vergnaud et al. (éds.), *Didactique et Acquisition des Connaissances Scientifiques*, (pp.149-164), Actes du Colloque de Sèvres, Mai 1987, Grenoble : La Pensée Sauvage.

DUVAL R. (1988) Pour une approche cognitive des problèmes de géométrie en termes de congruence, *Annales de Didactique et de Sciences Cognitives, 1*, 57-74.

FREUDENTHAL H. (1985) Notation mathématique, *Encyclopædia Universdlis*, 13, 144-150.

GLAESER G.(1971) *Mathématiques pour l'élève-professeur*, Paris : Hermann.

GRIZE J.-B. (1982) Réflexions pour une recherche sur l'argumentation, in J.-B.GRIZE (éd.) *De la logique à l'argumentation* (pp.133-142), Genève : Droz.

LABORDE C. (1982) *Langue naturelle et écriture symbolique : Deux codes en interaction dans l'enseignement mathématique*, Thèse de Doctorat d'Etat, Université Scientifique et Médicale, Grenoble.

MESQUITA A.L. (1989) *L'influence des aspects figuratifs dans l'argumentation des élèves en géométrie : Eléments pour une typologie*, Thèse de Doctorat, Université Louis Pasteur, Strasbourg.

PIAGET J. et INHELDER B. (1981) *La représentation de l'espace* (4e. éd.), Paris : PUF.

PIAGET J., INHELDER B. et SZEMINSKA A. (1973) *La géométrie spontannée de l'enfant* (2e. éd.), Paris : PUF.

PLUVINAGE F. et al. (1985) *Rapport sur l'expérimentation "Pédagogie Differenciéé" conduite en mathématiques au Collège d'Ostwald en 1984/85*, Pré-print, Université Louis Pasteur, Strasbourg.

THE EXPLANATION BY "EXAMPLE"

- For Establishing The Generality of Conjectures -

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ABSTRACT. *The previous researches about the recognition of proof have already pointed out the problem that students tend not to recognize deductive proof as being establishing the generality of propositions or conjectures. Concerning this problem, the aim of this paper is to propose the following statements as a hypothesis. If a student who only has empirical methods for validating conjectures, carries out the activities each of which has the following purpose and achieves each of them, then he can recognize and establish the generality of a conjecture by means of example.*

1: The construction of conjectures.
2: The validation of conjectures by empirical methods.
3: The identification of the series of interpretations and/or operations as one corresponding the schema.
4: The confirmation of the invariance of the schema.

0. Introduction

The teaching of proof aims that students can establish the generality of conjectures or propositions. But previous researches had not necessarily got the positive results about the students' notion of proof. This is one of reasons why students doesn't recognize the necessity and the appreciation of proof.

I set the following question as the research problem of this paper. How activities can lead a student who has only the empirical method for validating conjectures to recognize and establish the generality of a conjecture?.

Then, I propose the following hypothetical statements as the conclusion of this paper.

If a student who only has the empirical method for validating conjectures, carries out the activities each of which has the following purpose and achieves each of them, then he can recognize and establish the generality of a conjecture by means of example.

1: The construction of conjectures.
2: The validation of conjectures by empirical methods.
3: The identification of the series of interpretations and/or operations as one representing the schema.
4: The confirmation of the invariance of the schema.

1. The frame of study

"Explanation" means the discourse of a person who aims to establish the generality of conjecture. I set two levels of explanation.

Pragmatic explanation: the explanation which a person makes on the basis of concrete actions and its results. The representation has the particularity, but not necessarily uses language. The concrete action means calculation, measurement, etc.

Deductive explanation: the explanation which a person makes on the basis of the inference implying the necessary conclusion from

the mathematical properties and relations between them. The representation has no particularity, but uses language.

I use explanation by "example" in the following meaning.

<u>Explanation by "example":</u> the explanation with the series of interpretations and/or operations which a person shows together with the recognition of the generality of a conjecture, nevertheless depending on concrete actions. The series is corresponding to the schema of deductive explanation.

"Schema" is the system of mathematical properties and relations between them which deductive explanation has. Schema consists of two parts. One of them indicates the way to specialize the antecedent of a conjecture, the other indicates a chain of deductive inference from the specialized to the consequent of the conjecture. I call the former part "design" and the latter part "mood". "Example" (*1) in this paper has the possibility that a person can identify the series of interpretations and/or operations for the instance of the example as one corresponding to the mood, independent of the person, not only being the specialized of a case (*2).

For example, the following explanation is an explanation by "example" for the problem "What characteristics does the sum of 3 continuous numbers(for example,2,3,4) have?".

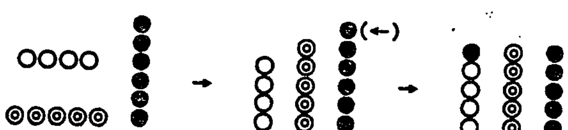

Fig.1-1 Fig.1-2 Fig.1-3

"When I represent (4,5,6) with marbles, I get this arrangement (Fig.1-1), next, arranging properly, I get this one (Fig.1-2). Then, by moving this marble, I get this (Fig.1-3) and new crosswise set of 3 marbles. the sum of 3 continuous number is a multiple of 3, because the arrangement is some crosswise sets of 3 marbles."

Some interpretations and operations are involved in this explanation. Each of them is corresponding to the mathematical properties and relations between them, each of which is involved in the schema of the deductive explanation for the previous problem. The series of interpretations and operation for making the middle arrangement (Fig.1-2) from the primitive arrangement (Fig.1-1) is corresponding to the design of the schema. The series for making the final arrangement (Fig.1-3) from the middle arrangement is corresponding to the mood of the schema. The former series makes us possible to find the latter series. Therefore, the series corresponding to the design of the schema is critical to make the explanation by "example".

Explanation by "example" has two characteristics common to pragmatic explanation.

1. A person carries out concrete actions.
2. The representation has the particularity.

Explanation by "example" has a characteristic common to deductive

explanation.

The series of explanation by "example" is corresponding to the schema of deductive explanation.

Namely, we can place explanation by "example" between pragmatic explanation and deductive explanation.

Deductive Explanation

The Exsistence of Schema

Explanation by "Example"

Carrying Out The Concrete Action | The Representation with The Particularity

Pragmatic Explanation

<u>*2. Four activities related to the generality of a conjecture*</u>

When a student in the level of pragmatic explanation faces to the problem requiring the validation of conjectures, he firstly needs a conjecture to be explained. So he needs to carry out the activity to achieve "the construction of conjectures".

Conjectures are ones for which a student expects to hold good but doesn't have enough convictions. There is a possibility that he rejects a correct conjecture will be rejected or accepts a wrong conjecture because of a little conviction . When a student can confirm the correctness of a conjecture by empirical methods, he can increase the validity of it by empirical methods. On the contrary, when he cannot confirm, he needs to reject. Therefore, he needs to carry out the activity to achieve "the validation of conjectures by empirical methods".

As far as the method for validating conjectures is empirical, a student only can show that the results are concurrent with the consequent of a conjecture in each case. When he faces to the question "Does the conjecture hold good in all case of domain which I have about the mathematical objects of the conjecture?", he tries to validate the empirical method for validating the conjecture. Then, he needs to objectify the empirical method itself.

There are 3 kinds of empirical methods at least. They are factual confirmation, crucial experiment(1), and the use of schema as the series of interpretations and/or operations. Especially, the last one is that a students gets the instance of the consequent of a conjecture by using the schema as the interpretations and/or operations to the instance of the antecedent of it. This empirical method "the use of schema as the series" has the mathematical properties and relations between them involved in schema as the method for validating the series. Therefore, a student can establish the validity of this empirical method by identifying the series of interpretations and/or operations as one corresponding to the schema. Then, I call this activity, not the activity to achieve "the objectification of the empirical method", but the activity to achieve "the identification of the series of interpretations and/or operations as one corresponding to the schema".

The domain which a student has about a mathematical object doesn't consist of the finite number of cases in general. So, A student tries to

confirm the invariance of the schema when he faces to the question "Can I apply the series to the instance of all cases?". Therefore, he needs to carry out the activity to achieve "the confirmation of the invariance of the schema". It is assumed that a student confirms the invariance of schema, when he confirms the followings.

1: It is possible to transform the case in which the series of interpretations and/or operations has been identified to all cases of domain by a constant rule.

2: When the series is made by representing schema in conformity with the instance of the transformed case, the series is applicable to the instance independent of the rule for transformation.

A student can generate the explanation by "example" in achieving 4 purposes above. Because these confirmation mean that he can necessarily get the instance corresponding to the antecedent of conjecture from the instance corresponding to the consequent in any case of his domain.

3. Actual states of four activities related to the generality of a conjecture

We observed the 6th grade students (Michiru & Yumi) in a public elementary school in Japan. The problem for the observation is " what characteristics does the sum of 3 continuous numbers(for example,2,3,4) have? Explain the reason". They are selected for the observation under teacher's suggestion so that they can discuss productively. The reason for pairing students is that they can express their thinking in the overt form as possible. They are intended to construct conjectures by the calculation of some specific cases, and to explain the generality of a conjecture by using marbles.

On account of space consideration, I describe the activity to achieve "the identification of the series of interpretations and/or operations as one corresponding the schema" and the activity to achieve "The confirmation of the invariance of the schema".

3.1 Actual state of the activity to achieve "the identification of the series of interpretations and/or operations as one corresponding to the schema"

Michiru & Yumi construct the conjecture "the sum of three continuous numbers is a multiple of 3" on the basis of some results by calculation, and validate it by the factual confirmation. Then, they set up the consequent of it as the purpose in the form of the interpretation for the arrangement of marbles "the arrangement is a set of 3 marbles" so that they answer the question "Does the conjecture hold good in any case?". They replaces the case of (4,5,6) into the arrangement of marbles from the lengthwise row corresponding to "4" in order with intending to achieve this purpose,. The arrangement by replacement is an instance of "example" (4,4+1,4+2).

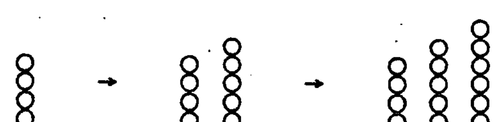

Fig.3-1

In counting the number of all marbles, Michiru counts in the lengthwise manner. On the contrary, Yumi counts in the crosswise manner from left to right. Though her activity might be contingent, we can consider that Yumi has changed the interpretation for the arrangement from the lengthwise rows to the crosswise rows on (or after) the counting.

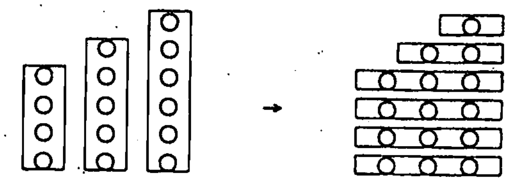

Fig.3-2

As they set up as the purpose the interpretation that the arrangement is a set of 3 marbles, the interpretation and operation that make another set of 3 crosswise marbles of 2 crosswise marbles and 1 marble, are indispensable to achieve the purpose. In fact, Yumi doesn't carry out this operation in "example" (4,4+1,4+2). But, it is Yumi that explains the conjecture in "example" (2,2+1,2+2) with marbles after observer's suggestion for the way of explanation. (see Fig.3-3, 3-4).

Fig.3-3 Fig.3-4

Therefore, It can be considered that Yumi has already recognized a constant series of interpretations and operatio::s to achieve the purpose in the instance of "example" (4,4+1,4+2). Namely Yumi validates the conjecture in (4,5,6) by the use of schema as the series of interpretations and/or operations.

After observer's suggestion, Michiru & Yumi immediately stop to think about (4,5,6) and start to think about (2,3,4). Though they have also taken dispensable operations to achieve the purpose (for example, counting all marbles before moving one marble and arranging the upper and lower sides and the left and right sides of marbles) in (4,4+1,4+2), they only take indispensable interpretations and operations in (2,3,4). We can consider that Yumi selects the indispensable interpretations and operations, and serializes them to achieve the purpose.

The series by the selection and the serialization is corresponding to the schema as following.

Interpretation 0:The arrangement of marbles is a set of 2 marbles, 3 marbles, and 4 marbles.

Operation 0:Arranging each row vertically in completing the lower side.

[Interpretation 1:The arrangement of marbles consists of the lengthwise row. The number of each row is 2, 2+1, 2+2.] (*3)

Operation1:Dividing the arrangement into three crosswise marbles.

Interpretation2:There are 2 sets of 3 crosswise marbles, 2 crosswise

marbles, and 1 marble.

Operation2:Making new 3 crosswise marbles by moving 1 marble to 2 crosswise ones.

[Interpretation3:There are 2 sets of 3 crosswise marbles and a new 3 crosswise marbles.]

Interpretation4:There are 3 sets of 3 crosswise marbles.

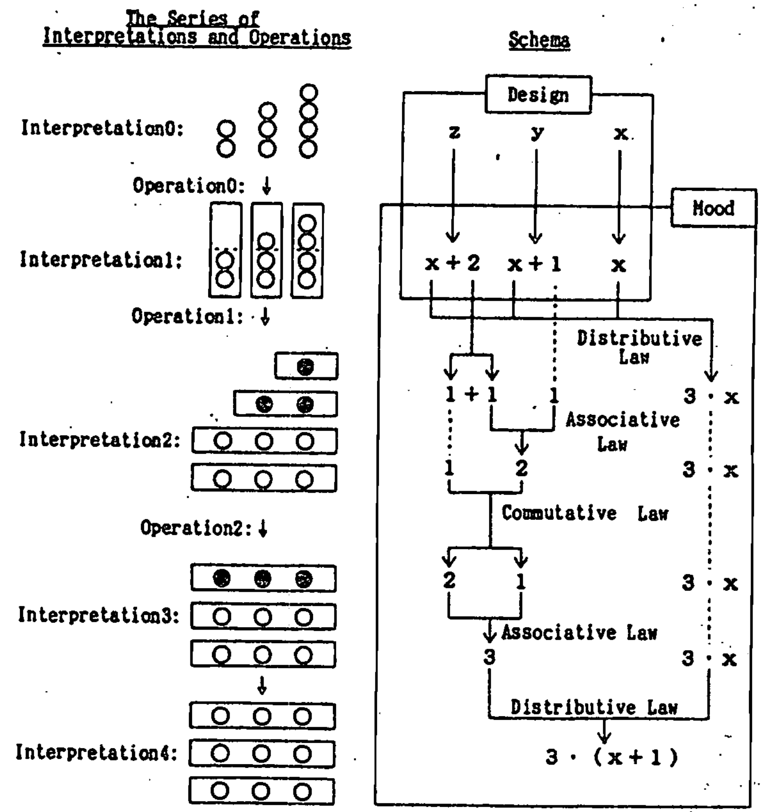

Fig.3-5

3.2 The activity to achieve "the confirmation of the invariance of the schema"

After identifying the series as one corresponding to the schema in the instance of (2,3,4), Michiru & Yumi face up to the problem that whether the series is applicable to any other case or not.

Yumi returns 1 marble of the 3*3 arrangement to where it was in (2,3,4). She decides the transformation T_2 :(2,3,4) $\longrightarrow$ (2+2,3+2,4+2) and attaches 2 marbles to each lengthwise row. Then, she confirms that she can apply the series representing the schema in the conformity with the

instance of (4,5,6) to the instance of (4,5,6). Additionally, Michiru & Yumi decide the other transformation T_1 :(4,5,6)$\rightarrow$(4+1,5+1,6+1) in cooperation, and attach 1 marble to each lengthwise row. Then they confirm that they can also apply the series representing the schema in the conformity with the instance of (5,6,7) to the instance of (5,6,7).

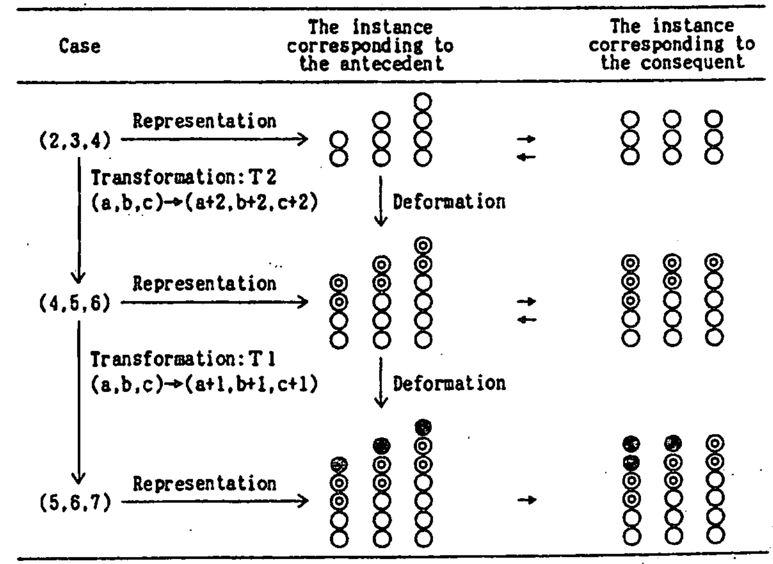

Fig.3-6

After the transformations T_1 & T_2, Michiru & Yumi immediatly start to generate the explanation by using the instance of "example" (2,2+1,2+2). When they stated to this activity, they faced to the previous question. Therefore, Starting to generate the explanation shows that they have resolved the question. Namely, we can consider that they recognize the applicability of the series of interpretations and operations to the instance of any other cases in their domain, when representing the schema in conformity with each instance.

Why they can recognize the applicablity of the series in any other cases by carrying out 2 kinds of transformations?. Yumi who carries out transformations T_2 & T_1 pays attention to the variable attribute of the instance of "example" (2,2+1,2+2) that each number of marbles is 2,3,and 4. Yumi's attention means that she finds the variable property of (2,3,4). Then, Yumi carries out transformations T_2 & T_1, and recognizes that she can apply the series of interpretations and operations representing the design of the schema to the instance of a case by the transformations. The series representing the design makes Yumi to recognize the instance as one which has the specific attribute "the relation of the number of lengthwise marbles is the number of the minimum, the number+1, the number+2". The attribute certifies the series representing the mood of the schema to be applicable to the instance.

Yumi specifies the attribute as the invariant. In addition, She induces the rule "each number changes by the same" as the consistent between the transformation T_1 and T_2. Then, it can be considered that she constructs $T = \{ T \mid T_n : (a,b,c) \rightarrow (a+n,b+n,c+n) \}$ and confirms the followings.

1: It is possible to transform (2,3,4) to all cases in Yumi's domain of 3 continuous number by the transformation involved in the family of transformation $T = \{ T \mid T_n : (a,b,c) \rightarrow (a+n,b+n,c+n) \}$.

2-a: The series representing the design gives the instance the specific attribute "the relation of the number of lengthwise marbles is the number of the minimum, the number +1, the number +2" ,independent of the rule of transformation.

2-b: The attribute certifies the series representing the mood to be applicable to the instance that the series representing the design makes.

4.Questions

1: What characteristics can we find about the explanation by "example" in relation to the knowledge and the language used?.

2: Can we decompose one activity into some activities?.

3: Can we identify the activities to lead a student to generate the deductive explanation, if he can generate only the pragmatic explanation and the explanation by "example" ?.

Remarks:

*1: Case is the component of a person's domain about a mathematical object as mathematical entities. Example is the case to which special properties are attached. Instance is the physical entities representing case or example in the previous sense.

*2: The reason why I cannot use the term "Generic Example" for "example" is following. It highly depends on a person who uses the example that whether a example is generic or not.

*3: The observer couldn't identify the interpretation or the operation in [ ] as the student's overt behavior. But they seem to be necessary so that the other interpretations and operations have a logical consistency.

Reference:

1:Balacheff,N.(1988).A study of pupil's proving processes at the junior high school level. Paper presented at the joint international conference of 66th annual meeting of National Council of Teachers of Mathematics and University of Chicago School Mathematics Project,Chicago.

2:Mason,J. & Pimm,D.(1984).Generic Example:Seeing the General in the Particular. Educational Studies in Mathematics 15(1984).277-289

3:Semadeni,Z.(1984).Action Proof in Primary Mathematics Teaching and in Teacher Training. FLM 4.1.1984.32-34.

4:Walther,G.(1986).Illuminating examples-an aspect of simplification in the teaching of mathematics. Int.J.Math.Educ.Sci.Technol.Vol.17, No.3,263-273

TEACHERS' ATTITUDES TOWARDS MATHEMATICS AND MATHEMATICS TEACHING: PERSPECTIVES ACROSS TWO COUNTRIES

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This paper[1] reports the findings of a study to investigate primary teachers' attitudes towards mathematics and mathematics teaching. The study took place in England and Portugal, and the results are interpreted from a vantage point gained by this cross-cultural perspective.

During the last decade or so, there has been a resurgent interest in researching teachers' attitudes towards mathematics and towards mathematics teaching. This might be due to a growing influence that teachers' attitudes have in their instructional practices, and in particular to the ways they are able to influence the course of educational change. The purpose of this paper is to provide further information on these issues by adapting a vantage point which spans two educational cultures.

The aims of the research were twofold: 1) to provide information on primary teachers' attitudes towards mathematics and towards mathematics teaching both in England and Portugal; and 2) to examine factors, and in particular culturally-bound ones, that might account for these attitudes[2].

Conceptualising Teachers' Attitudes

The study of attitude is problematic. One of the sources of difficulty arises from the fact that the term has been defined in a variety of ways, and there is no uniformity of views on what attitudes are and how they change (see for example Hart, 1989 and McLeod, 1989 for elaboration of this point in relation to mathematics education). The state of present attitude theory is such that there is little strong rationale for adopting one definition and rejecting other. The point of view we adopted had its origin in the definition of attitude presented by Petty and Cacciopo (1986):

> Attitudes are general evaluations people hold in regard to themselves, other people, objects, and issues. These general evaluations can be based on a variety of behavioural, affective, and cognitive experiences, and are capable of influencing or guiding behavioural, affective, and cognitive processes (p. 127).

Especially important for the context of the study is that the above definition is comprehensive in the sense that attitudes may focus on people, objects and issues. Moreover, attitudes have the inherent property of being differentially manifested along a range of dimensions or constructs rather than a single one. The definition can therefore be useful in addressing a variety of situations related to mathematics and mathematics teaching. For example the expression "attitudes towards mathematics" may be used to refer not only to the degree of one's liking or disliking for mathematics, but also the confidence

[1] I am grateful to my supervisor Dr Richard Noss for his helpful comments on an earlier draft of this paper.

[2] The work reported here forms part of a broader study undertaken in England and Portugal with the purpose of investigating primary teachers' attitudes towards mathematics and mathematics teaching, and how these attitudes affected and were affected by their participation in a Logo based mathematical in-service course.

in one's ability to deal with mathematics, as well as what one thinks mathematics is about.

For the purposes of this study we conceptualised teachers' attitudes *towards mathematics* in terms of four constructs:

- –*Nature of Mathematics* (teachers' perceptions about mathematics);
- –*Mathematics in School* (teachers' views of the subject taught in school);
- –*Mathematics and Oneself* (the way teachers feel personally towards mathematics in terms of both enjoyment and confidence);
- –*Value of Mathematics* (teachers' perceptions about the importance and usefulness of mathematics in society).

Teachers' attitudes towards the *teaching of mathematics* were captured into four separate but interrelated categories:

- –*Aims of Teaching Mathematics* (teachers' opinions about the aims of teaching mathematics in primary school);
- –*Nature of Mathematics Learning* (teachers' theoretical principles about how the learning of mathematics by the individual child better takes place);
- –*Classroom Context for the Learning of Mathematics* (teachers' intentions of behaviour concerning the classroom atmosphere in relation to pupils' mathematical learning);
- –*Computers in Mathematics Learning* (teachers' opinions about the role of computers in children's mathematical learning).

In addition to conceptualising teachers' attitudes we also developed a framework to help us to understand how those attitudes have formed and evolved. After a review of relevant literature, a number of variables which might account for teachers' attitudes towards mathematics and its teaching were selected. In the event, we considered a total of 15 core variables clustered around three main classes:

Teachers' personal characteristics	*Teachers' Academic and Professional Experiences*	*Context Variables*
Sex	Highest qualifications in mathematics	Type of school
Age	Past experiences with mathematics	Size of school
	Initial teaching training qualificatons	School geographical situation
	Furtherteaching training qualifications	Age group taught
	Years of primary teaching experience	
	Other teaching experience	
	Posts of responsibility related to teaching	
	Use of computers in mathematics lessons	
	Attendance to in-service courses in maths	

Methodology

In order to satrisfy the aims of the research, it was necessary to assess the attitudes of a reasonably large sample of teachers in each country. This, in turn, suggested that from the variety of approaches existing for assessing people's attitudes a survey[3] mode of data collection was appropriate.

The Questionnaire. In the absence of existing instruments which sought for the kind of information that we were interested in, it was necessary to develop aquestionnaire specifically for the purpose of the research. After a series of pilots, the questionnaire which was finally developed consisted of three sections. The first two sections included 40 items designed to assess teachers' attitudes on eight subscales which corresponded to the constructs

[3] The surveys were supported in part by Suffolk County Council, in England, and by Direcção Escolar do Distrito de Viseu, in Portugal. We gratefully acknowledge their assistance in the questionnaires distribution and collection.

described earlier. The last section of the questionnaire sought for factual information about teachers' background variables.

Each subscale consisted of four to six items to which the teachers responded on a 5-point scale by indicating *strongly agree, agree, uncertain, disagree, or strongly disagree*. It should be noted that with the exception of the items exploring teachers' opinions about the aims of teaching mathematics, all the remaining items were selected in order to form different Likert type attitude scales. While bearing this in mind, these items were choosen to providing per se useful information about teachers' attitudes too. Examples of the items on the subscales are given below:

Subscale	Example
Nature of Mathematics	Maths is consistent, certain, and free of ambiguities
Mathematics in School	Maths is as creative a subject as art or music
Mathematics and Oneself	I feel a sense of insecurity whendealing with maths
Value of Mthematics	There is little need for more than very elementary maths in most jobs
Aims of Teaching Mathematics	The main able to teach mathematics is to enable pupils to appreciate and enjoy it for its own sake
Nature of Learning Mathematics	Expecting pupils to discover mathematical ideas by hemselves is unreasonable
Classroom Context for the Learning Mathematics	As a rule, in my maths lessons, I encourage pupils to *of* work cooperatively
Computers in Maths Lessons	Computers are specially useful in allowing children to practise mathematical skills

Following a technique similar to that developed by Likert (1967), items in each subscale were classified as being favourable and unfavourable. Answers to favourable items were scored from 1 to 5, and in the reverse order for unfavourable items. By summing up across all the items included in a subscale, composite measures for each teacher and for each subscale were obtained.

Validity and reliability of the subscales developed were assessed making use of the results of the study itself. Here, we can only note that the reliability of the scales as measured by the Cronbach coefficient varied between .23 and .76. Interestingly, the reliability of the scales was smaller and in some cases considerably smaller in Portugal than in England. There are a number of possible explanations for this: we offer one, namely the use of attitude scales for comparison between countries raises issues of sensitivity. Thus, although careful attention was given to translating the questionnaire into Portuguese (the questionnaire was initially developed in English), there were likely to be differences in the extent to which a term was applicable in the two countries.

The Participants. The questionnaire was administered to all primary teachers in a semi-rural district of Portugal, and a comparable one in England. Data collection in England spanned the October-January period of the 1987/88 academic year, and in Portugal the February-March period of the same academic year.

A total of 586 questionnaires in England and 1680 questionnaires in Portugal were returned in the due time. These figures corresponded to approximately 20% and 84% of the intended populations in England and Portugal. A possible source of the different rate of answers is a certain fatigue among English teachers as a result of participating in various surveys compounded with a tendency to comply with Local Educational Authorities

initiatives among Portuguese teachers. It should be noted, however, that in both countries in those aspects in which comparison was possible, the profile of those who returned the questionnaire was similar to that of those provided by the official statistics.

Data Analysis Considerations. Data were analysed by making use of the Statistical Package for the Social Sciences (SPSS). The two data sets from the two countries were analysed separately and compared. The analysis of respondents' attitudes was carried out via two methods. The first consisted of the analysis of the distributions of actual frequencies and percentages of respondents' answers to the individual items in terms of the five alternative options (from strongly disagree to strongly agree). Missing answers and written comments to individual items were also noted. This was considered specially important in assessing the validity of the items. The next step was to find for each composite score the frequency charts, and the commonly-used measures of tendency (mean, median, mode) and of spread (minimum, maximum, and standard deviation). As already mentioned, not all of the items originally included in any subscale were used to calculate the single summary score. When the number of retained items contributing to a subscale fell below three, respondents' answers were analysed only at item level. As far as teachers' opinions about the aims of teaching mathematics are concerned, these were analysed both in terms of actual frequencies and percentages, and in terms of their relative hierarchical ordering.

In comparing the results across the two countries, special attention was given to the fact that the attitude scales that were used to assess teachers' attitudes are not infallible yardsticks and that cultural differences are of major significance in interpreting scales scores. Therefore, for the purpose of comparing the teachers' answers to individual items, the initial 5-point scale was collapsed into a 3-point scale (by concatenating the categories "strongly agree" and "agree" on the one hand, and the categories "strongly disagree" and " disagree" on the other hand). The chi-square test was used to assess the extent to which the distributions in the two populations were significantly different ($p<.001$). We also resisted the temptation of interpreting different mean attitude subscales scores in the two countries as necessarily meaning different attitudes. Instead, a simple form of profile analysis was carried out (Morrison, 1978) aimed at highlighting whether or not the profiles of mean scores in the different attitude scales for the two populations were similar. It should be noted that only five of the initially developed composite measures were used: *Nature of Mathematics*, *Mathematics in school*, *Mathematics and Oneself*, *Nature of Learning Mathematics* and *Computers in Mathematics Learning*. For the purpose of between-countries comparison, total scores were computed taking into consideration only the items which were found to differentiate well in both countries. When the number of common descriminating items contributing to an attitude scale fell below three, the comparison was performed at item level.

Selected Findings

In our presentation of the results and discussion, we concentrate upon the findings related to the comparison of teachers' attitudes in the two countries.

First of all, it is worth noting that there was a considerable difference in the factors which seemed to account for the variations of teachers' attitudes within each country. That is, most of the factors seemed to be specific to the country. In this regard, it should be noted that in Portugal the variability of attitudes explained by the factors investigated was extremely small that the issue hardly deserves any further consideration. This fact can be attributed to both the relative low reliability of the scales in Portugal, and the relative homogeneity of Portuguese teachers' attitudes. In England the variability of attitudes explained by the factors investigated varied between 7% and 22%. This may mean that the importance of the factors investigated was not as significant as expected, and that other relevant variables associated with teachers' attitudes were left out. This finding led us in what follows not to raise issues of comparability of teachers' attitudes in both countries in terms of factors considered, and on the contrary to adrress them in light of cultural and contextual differences.

Variation of Teachers' attitudes towards mathematics and mathematics teaching Teachers in both countries varied significantly in their attitude scores on each of the five attitude subscales that were used for comparative purposes. However, the differences of attitudes were more pronounced among the English teachers than among the Portuguese ones. We might offer three speculative explanations for the lower variability of opinions among Portuguese teachers. First, the existence of a national curriculum in Portugal may contribute to less variation among the Portuguese teachers. At the time of this research, there was no National Curriculum in the UK. Second, a very standardised type of teacher-training programme, as well as the lack of in-service opportunities may well lead to more stereotyped attitudes among the Portuguese teachers than among the English ones. Third, it is normally accepted that people in industrialised countries are more likely to value individualism and individual differences than in less industrialised ones. This may well lead to a lower degree of socialisation of teachers in England than in Portugal and therefore to greater variability of opinions.

Comparison between Teachers' personal feelings towards mathematics and their views about the subject The patterns of the profiles of mean scores of the subscales related to teachers' attitudes towards mathematics were quite different for the two populations. Of special interest is the crossover in the two profiles. While the Portuguese teachers scored higher in the *Mathematics and Myself* scale than the English teachers, the later scored higher in the *Nature of Mathematics* and *Mathematics in School* scales.

The Portuguese teachers tended to affirm to a greater extent the notion that mathematics is an absolute and rigorous field of endeavour. That Portuguese teachers' views about mathematics or school mathematics differ radically, at least in some aspects, from their colleagues in England was amply evidenced by their answers to the item: "Each mathematical problem has only one best way to get the solution". The item that had been initially included in the *Mathematics in School* scale was discarded in both countries for not discriminating well but for rather different reasons. Whereas in Portugal more than 80% of the respondents endorsed this view, in England about 90% of the teachers disagreed with it.

In turn, the English teachers were more likely to express lack of self-confidence and enjoyment with mathematics than their counterparts in Portugal. Are teachers who express an absolutist view of mathematics more likely to enjoy and feel confident with the subject than those who see the subject as a creative field? The analyses undertaken within each country at the individual level did not point to any kind of relationship between teachers' views of the subject and their personal feelings. Thus, the contrasted relationship found between the two countries has to be seen in light of other factors, and three come to mind.

First, the fact that the teachers in England appear to hold a less absolutist and formalistic perspective of mathematics than the teachers in Portugal may reflect the the state of school mathematics in the former. Indeed, what may be called the "English primary curriculum" has been relatively broad and investigative compared with the traditional, and formal approach dominant in Portugal. In other words, the English teachers' views of mathematics might be seen as an extension of their views about school mathematics. In Portugal, the changes that have taken place within the primary mathematics curriculum might have left teachers in a state of confusion and uncertainty rather than influencing their views about mathematics. It is possible that the low consistency shown by the teachers in Portugal in answering to the items included in scale is an expression of such uncertainty and confusion. Indeed, this is a potential risk when educational change is centrally determined, and governement and its agencies fail to provide appropriate information and advice to teachers on how to go about it.

The other two factors have to do with the fact that teachers in Portugal tended to show that they were more fond of mathematics and feel more confident with the subject than the teachers in England. One of the factors lies in the apparent tendency to give socially-desirable answers. Data from this study suggest that the Portuguese respondents were more likely to give socially-desirable answers than the English ones. This would mirror the cultural/social differences between the two countries, and in particular the atmosphere among Portugues primary teachers at the time of the survey. Indeed, new developments in the system of primary teacher-training and appraisal were taking place in Portugal and the "old" teachers were likely be willing to appear boastful about their professional prowess. As liking mathematics and being good at it are widely perceived as socially desirable attributes, it is possible that the Portuguese teachers tended to endorse such sentiments. The other factor is related to teachers' background in mathematics. In England, little more than 10% of the teachers participating in the survey reported that they had an A-level[4] in mathematics, whereas in Portugal the percentage of teachers who reported equivalent mathematics qualifications was considerably higher (about 25%). It may well be assumed that the higher their qualifications in mathematics, the more likely it is that a person can feel confident with the subject and enjoy it (indeed, further analyses undertaken within this study seemed to confirm this relationship). However, one should not take the relationship between mathematics qualifications and personal feelings as a purely causal one. For example, we may speculate that people

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might well only take A-level in mathematics because they already enjoy and feel confident with the subject.

Comparison between teachers' attitudes towards the teaching of mathematics and their views about the subject The pattern of scores in the two subscales, *Nature of Mathematics* and *Learning of Mathematics*, were similar in England and in Portugal. This is an interesting finding, in that a similar outcome is produced in spite of obvious differences in contextual variables (e.g. type of teacher training, type of schools) in both countries. One may interpret this result by stating that in both countries teachers' views about how children best learn mathematics appear to reflect mainly their underlying views about mathematics. That is, teachers who have an absolutist view about mathematics tend to see children's mathematics learning mainly as reception of mathematical knowledge and practice of basic skills, whereas teachers with a more open perspective of the subject are more likely to see children able of constructing their knowledge and learning in relation to understanding and problem solving. The same kind of relationship was found within each country. If a teacher sees mathematics as a set of unquestionable truths it is unlikely that there will be room for children to construct their own knowledge and learning mathematics becomes essentially a matter of memorising rules and practising them until they master them. But to see the relationship as a causal one is problematic. It may well be that the relationship between teachers' views about mathematics and about how children learn mathematics is mediated through other variables such as the school mathematics curriculum and fashionable claims about children's learning in general. Because of the complex nature of the issues addressed by these two scales, this question of such relationship is unlikely to be answered by the use of a paper-and-pencil research instrument. Indeed, what appears to be inconsistent answers to different items left us with room for a possible fruitful area of investigation. For example, in Portugal more than 85% of the teachers endorsed the item expressing that "With little guidance most pupils should be able to discover most mathematical ideas for themselves", and at the same time about 70% agreed with the statement that "pupils learn maths better by attending to teacher's explanations than by being left to make sense of things for themselves".

A comparison of teachers' answers to the items concerning the aims of teaching mathematics points to some similarities but also to marked differences as well. For example, in both countries the teachers accorded in emphazing above all the aim expressing the use of maths in "everyday-life" situations, and in giving minor importance to the statement "The main goal of teaching mathematics is to produce students who can perform the tasks especified in the curriculum". However, whereas in Portugal 38% of the teachers agreed with the that curriculum related aim, in England the percentage of teachers endorsing that view was little more than 10%. This sharp difference may be interpreted in light of the different educational systems in the two countries, and in particular due to the existing national curriculum in Portugal and the lack of one in UK. Two other important differences concerned the aims which stressed primary mathematics as a foundation for further mathematics and that of learning mathematics for its

own sake. These differences, however, are more likely to be related to teachers' views of mathematics than directly related to differences between the two countries. For example, the fact that the former of these aims seemed to be of considerable importance for the Portuguese teachers (the second in ranking, with about 86% of respondents endorsing that view) and of relatively little importance for the English teachers (the last but one in the ranking order, with only 50% of the respondents agreeing with it) may be regarded as congruent with a perspective of mathematics as an hierarchical and absolutist body of knowledge to be transmitted.

There were also substantial differences between the English and Portuguese teachers' responses to the items included in the *Context for the Learning of Mathematics Scale.* Reflecting "national" trends of opinion, two items were discarded from the scale in England and one item was discarded in Portugal in such a way that composite measures could not be used for comparison between countries. For example, most teachers in Portugal (about 83%) appeared to agree that they "demonstrate procedures and methods for performing mathematical tasks", whereas in England little more than 40% of the teachers endorsed that view. One possible interpretation of these results is that teachers' intentions of behaviour concerning the classroom atmosphere in relation to pupils' acquisition of mathematical knowledge reflect more of the social context in which the school operates (and probably that of the school) than their views about mathematics. That is, the differences between the countries with regard to the results under consideration may be said to be asociated to school related factors such as size, school learning resources, and administration. are likely to have a great impact on the general atmosphere of the school and this in turn is likely to affect teachers' intentions of behaviour about the classroom context for the learning of mathematics. In addition, cultural and contextual factors such as the strong value put on the ideals of democracy emerging in Portugal after 1974 may have brought about opinions among the Portuguese teachers such as "letting pupils to do whatever they want" and "teaching all the pupils in the same way".

References

Hart, L.E. (1989). Describing the Affective Domain: Saying what we mean. In D. B. McLeod & V. M. Adams (Ed.), Affect and Mathematical Problem Solving. (pp.37-45). New York: Springer-Verlag.

Likert, R. (1967). The Method of Constructing an Attitude Scale. In M. Fishbein (Ed.), Attitude Theory and Measurement. (pp.90-95). New York: John Wiley & Sons. (Excerpted from the Appendix of "A Technique for the Measurement of Attitudes," Archives of Psychology, 1932, No. 140, 44-53).

McLeod, D. B. (1989). Beliefs, Attitudes, and Emotions: New Views of Affect. In D. B. McLeod & V. M. Adams (Ed.), Affect and Mathematical Problem Solving. (pp.245-258). New York: Springer-Verlag.

Morrison, D. F. (1978). Multivariate Statistical Methods (2nd ed.). Auckland: McGraw-Hill Book Company.

Petty, R. E. & Cacioppo, J. T. (1986). The Elaboration Likelihood Model of Persuasion. Advances in Experimental Psychology, 19, 123-201

WORLD PROBLEMS - THE CONSTRUCTION OF MULTIPLICATIVE STRUCTURES

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SUMMARY

In our research we try to analyse the relationship between the construction of mental representation, strategies and correct answers when a group of 45 subjects, from a primary school in Portugal, (15 from the 2nd grade, 15 from the 3st grade and 15 from the 4st grade), have to solve 4 word problems involving the construction of multiplicative structures. The results show a direct relationship between a correct mental representation and a correct solution but they do not show a very clear relationship between a correct (or incorrect) mental representation and certain kinds of strategies.

INTRODUCTION

Riley, Greeno and Heller (1983) think, that three different features must be taken into account when the child solves a word problem: 1. the important data of the problem. 2. the construction of a correct mental representation(1) . 3. the choice of a correct strategy.(2) When a child tries to solve an arithmetic word problem, it can happen that he constructs a correct mental representation but fails in the choice of the strategy; it can also happen that he constructs an incorrect mental representation, and according to this one, he chooses an incorrect or correct strategy. In all this cases, normally, the solution will be incorrect. Indeed, to solve a word problem the child has to choose the important data, to organise a correct mental representation and, further, he needs to find a strategy and to employ it without fails.

In our report we would deal with word problems involving multiplication, and although different authors set up different classifications of this types of word problems (Greer 1987), in this study we choose the Vergnaud's classification which is, in our opinion, shorter and clearer than others. According to this author (1981) we have three different types of multiplicative word problems: A. Isomorphism of measures. B. One measure. C. Product of measures.

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(1) Mental representation. This is the child's organisation and understanding of the semantic structure of the data of the word problem.

(2) Strategy - This is the way the child finds out to solve the problem. (To choose an operation or operations and to make all the calculations).

In turn, the English teachers were more likely to express lack of self-confidence and enjoyment with mathematics than their counterparts in Portugal. Are teachers who express an absolutist view of mathematics more likely to enjoy and feel confident with the subject than those who see the subject as a creative field? The analyses undertaken within each country at the individual level did not point to any kind of relationship between teachers' views of the subject and their personal feelings. Thus, the contrasted relationship found between the two countries has to be seen in light of other factors, and three come to mind.

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The other two factors have to do with the fact that teachers in Portugal tended to show that they were more fond of mathematics and feel more confident with the subject than the teachers in England. One of the factors lies in the apparent tendency to give socially-desirable answers. Data from this study suggest that the Portuguese respondents were more likely to give socially-desirable answers than the English ones. This would mirror the cultural/social differences between the two countries, and in particular the atmosphere among Portugues primary teachers at the time of the survey. Indeed, new developments in the system of primary teacher-training and appraisal were taking place in Portugal and the "old" teachers were likely be willing to appear boastful about their professional prowess. As liking mathematics and being good at it are widely perceived as socially desirable attributes, it is possible that the Portuguese teachers tended to endorse such sentiments. The other factor is related to teachers' background in mathematics. In England, little more than 10% of the teachers participating in the survey reported that they had an A-level[4] in mathematics, whereas in Portugal the percentage of teachers who reported equivalent mathematics qualifications was considerably higher (about 25%). It may well be assumed that the higher their qualifications in mathematics, the more likely it is that a person can feel confident with the subject and enjoy it (indeed, further analyses undertaken within this study seemed to confirm this relationship). However, one should not take the relationship between mathematics qualifications and personal feelings as a purely causal one. For example, we may speculate that people might well only take A-level in mathematics because they already enjoy and feel confident with the subject.

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Comparison between teachers' attitudes towards the teaching of mathematics and their views about the subject The pattern of scores in the two subscales, *Nature of Mathematics* and *Learning of Mathematics*, were similar in England and in Portugal. This is an interesting finding, in that a similar outcome is produced in spite of obvious differences in contextual variables (e.g. type of teacher training, type of schools) in both countries. One may interpret this result by stating that in both countries teachers' views about how children best learn mathematics appear to reflect mainly their underlying views about mathematics. That is, teachers who have an absolutist view about mathematics tend to see children's mathematics learning mainly as reception of mathematical knowledge and practice of basic skills, whereas teachers with a more open perspective of the subject are more likely to see children able of constructing their knowledge and learning in relation to understanding and problem solving. The same kind of relationship was found within each country. If a teacher sees mathematics as a set of unquestionable truths it is unlikely that there will be room for children to construct their own knowledge and learning mathematics becomes essentially a matter of memorising rules and practising them until they master them. But to see the relationship as a causal one is problematic. It may well be that the relationship between teachers' views about mathematics and about how children learn mathematics is mediated through other variables such as the school mathematics curriculum and fashionable claims about children's learning in general. Because of the complex nature of the issues addressed by these two scales, this question of such relationship is unlikely to be answered by the use of a paper-and-pencil research instrument. Indeed, what appears to be inconsistent answers to different items left us with room for a possible fruitful area of investigation. For example, in Portugal more than 85% of the teachers endorsed the item expressing that "With little guidance most pupils should be able to discover most mathematical ideas for themselves", and at the same time about 70% agreed with the statement that "pupils learn maths better by attending to teacher's explanations than by being left to make sense of things for themselves".

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OBJECTIVES

In this research we have two different objectives. The first one is to analyse what kind of mental representation (correct or incorrect) is constructed by the children, on the 2nd, 3st and 4st grade of a Primary school in Portugal, according to Vergnaud's classification of multiplicative word problems. The second one is to analyse the relationship between mental representation, strategy and correct answer within each word problem.

METHODOLOGY

We have 4 multiplicative word problems to present to the children.

ISOMORPHISM OF MEASURES

1. Exchange Rate:

Tommy's father gave him 2 pesetas when he was coming back from Spain and he told him that it was like to have 6 escudos. Later on, he gave Tommy 4 pesetas. How many escudos can Tommy get for this 4 pesetas?

2. Multiple groups: Mary's mather brought from the supermarket 4 bags with 4 yogurts in each of them. How many yogurts are there altogether?

ONE MEASURE

3. Tommy bought a candy for 5 escudos. The shop-assistant told him that a packet of candies costs 5 times more. How much does the packet cost?

PRODUCT OF MEASURES

4. One girl has 4 blouses and 3 skirts. How many different combinations of blouse and skirt can she dress?

All this problems are presented randomly.

1. The experimenter reads the problem. 2. The child reads the problem and explains it to the experimenter. 3. The experimenter gives him a concrete material, adapted to each

problem, and asks him to represent with it the word problem(3) . 4. The experimenter asks the child to give the solution and to explain verbally how he found it. (A similar methodology was used by De Corte and Verschaffel with word problems involving addition and subtraction).

Our sample is composed by 15 children of each grade (2nd, 3st and 4st). The average age is respectively; 7,4; 8,6; 9,3.

RESULTS

2nd Grade - Isomorphism of measures

1. Exchange Rate - 4 children have a correct mental representation of this word problem (they show it using the concrete material). 2 have an additive strategy (2 pesetas + 2 pesetas = 4 pesetas so 6 escudos + 6 escudos = 12 escudos) and 2 a multiplicative one (2 pesetas X 2 = 4 pesetas so 6 escudos X 2 = 12 escudos). 11 children show an incorrect mental representation; 3 of them think that 1 peseta is 6 escudos; they use an additive strategy; 7 children organise their mental representation considering 4 pesetas = 4 escudos. They use a counting strategy (1,2,3,4 escudos or 1,2,3,4,5,6,7,8,9,10 escudos). 1 child take into account all the coins placed over table and, using a counting strategy, finds, by chance, the correct solution.

2. Multiple Groups. 9 children have a correct mental representation of this word problem (to place the yogurts into the bags). 3 use a counting strategy (1.........16). 5 have and additive strategy (4+4+4+4=16) and 1 has a multiplicative one (4X4=16). 6 children have an incorrect mental representation and all of them use a counting strategy.

ONE MEASURE

3. Keyword problem. 8 children have a correct mental representation (they construct 5 groups of 5 candies each)--, 6 use an additive strategy (5+5+5+5+5 =25) and 2 a multiplicative one (5X5=25). 6 have a incorrect mental representation; 2 of them use a counting strategy (1,2,3,4,5) and 4 an additive one (5+5=10).

(3) The concrete material can help the child in the construction of a correct mental representation and can also help the experimenter to analyse it.

PRODUCT OF MEASURES

4. Combination. 7 children have a correct mental representation. (They accept that it is possible to combine blouses and skirts in different ways). The counting strategy used by them is not systematique and so, only 1 child, by change, solve the problem correctely. 8 children have an incorrect mental representation. (They dont' accept that it would be possible to make more than 3 combinations). They also use a counting strategy.

3^{st} GRADE

ISOMORPHISM OF MEASURES

1. Exchange Rate: 8 children have a correct mental representation of this word problem; 5 use an additive strategy and 3 a multiplicative one. 6 children have an incorrect mental representation (1 peseta = 6 escudos); 1 of them uses an additive strategy and 5 a multiplicative one. The last child has also an incorrect mental representation (all coins placed over the table are escudos) and using a counting strategy, by chance, he solves the problem correctely.

2. Multiple groups: 14 children have a correct mental representation of this word problem; 5 use an additive strategy and 9 a multiplicative one. 1 child has an incorrect mental representation and he uses a counting strategy.

ONE MEASURE

3. Keyword problem: 10 children have a correct mental representation and 9 of them use a multiplicative strategy to solve the problem; the other one uses an additive strategy. 4 children have an incorrect mental representation; all of them use an additive strategy and 1 tell us, without explanation, that the result is 50 escudos.

PRODUCT OF MEASURES

4. Combination: 5 children have a correct mental representation of this problem. They use a counting strategy incorrectely and only 1 subject, by chance, finds the solution. 10 children have a incorrect mental representation but they also use a counting strategy.

4^{st} GRADE

ISOMORPHISM OF MEASURES

1. Exchange Rate: 11 children have a correct mental representation of this word problem; 10 use an additive strategy and 1 a multiplicative one. 4 children have an incorrect mental representation (1 peseta = 6 escudos); 2 of them use an additive strategy and 2 a multiplicative one.

2. Multiple groups. All children have a correct mental representation. 1 uses a counting strategy, 9 an additive one and 5 a multiplicative one.

ONE MEASURE

3 . Keyword problem. All subjects have a correct mental representation of this word problem; 2 of them use an additive strategy (they make a mistake in the calculation of the solution) and 13 use a multiplicative one.

PRODUCT OF MEASURES

4. Combination. 12 children have a correct mental representation. 1 of them uses the counting strategy correctely and finds the solution, by chance. 3 children have an incorrect mental representation and also use the counting strategy.

In this tables we can see, as a whole, the results found by subjects in each word problem.

EXCHANGE RATE PROBLEM

	M. Representation		Strategy			Answer	
	C.	Inc.	Count.	Add.	Mult.	C.	Inc.
2^{nd} class	4 - 26%	11	8	5	2	5 - 33%	10
3^{st} class	8 - 53%	7	1	6	8	9 - 60%	6
4^{st} class	11 - 73%	4	-	12	3	11 - 73%	4

MULTIPLE GROUPS PROBLEM

	M. Representation		Strategy			Answer	
	C.	Inc.	Count.	Add.	Mult.	C.	Inc.
2^{nd} class	9 - 60%	6	9	5	1	9 - 60%	6
3^{st} class	14 - 93%	1	1	5	9	14 - 93%	1
4^{st} class	15 - 100%	-	1	9	5	14 - 93%	1

KEYWORD PROBLEM

	M. Representation		Strategy			Answer	
	C.	Inc.	Count.	Add.	Mult.	C.	Inc.
2^{nd} class (1)	8 - 53%	6	2	10	2	7 - 46%	7
3^{st} class	10 - 66%	4	-	5	9	10 - 66%	5 (2)
4^{st} class	15 - 100%	-	-	2	13	13 - 86%	2

(1) One subject refused to answer this word problem
(2) One subject gave us an incorrect answer without any explanation.

COMBINATION PROBLEM

	M. Representation		Strategy		Answer	
	C.	Inc.	Count.	Add.	C.	Inc.
2^{nd} class	7 - 46%	8	15	-	1 - 6%	14
3^{st} class	5 - 33%	10	15	-	1 - 6%	14
4^{st} class	12 - 80%	3	15	-	4 - 26%	11

Discussion

1. If we take into accout all the word problems presented to our subjects we can say that the multiple groups problem and the keyword problem show the highest percentage of correct answers in the 2^{nd}, 3^{st} and 4^{st} grades. The combination problem, on the contrary, shows the highest percentage of incorrect answers between the subjects. In all the word problems, the percentage of correct answers increases from the 2^{nd} to the 4^{st} grade.

2. It was also noticed by us that 2 subjects solved, by chance, the exchange rate problem, and 5 did it in the combination problem. It means that sometimes it can happen that the children find the solution without any understanding of the word problem or without using a strategy in a perfect way. Pedagogical implications of this conclusion are obvious; teachers have to be aware that a correct answer does not necessarily imply a real comprehension of the problem. In same cases, the contrary can also happen; children can give us an incorrect answer and to have construct a correct mental representation and choose a correct strategy; the fail can be found on the calculation (as we have seen in the keyword problem).

3. When we compare the strategies used by the children to solve all the word problems we notice that the counting strategies almost disappear on the 3^{th} and 4^{th} grade. (except in the combination problem). If we analyse the exchange rate problem and the multiple groups problem we can see that the multiplicative strategy is more commum between the children of the 3^{th} grade than between the subjects of the 4^{th} grade. In this last group the additive strategy is used more often. (this results must be taken with precautions because we have a very small sample). Our explanation for this results is related with the arithmetic curriculum in primary school in Portugal. The children learn, during the 3^{st} grade, the multiplicative calculations and so, this structure is present to them at this moment and can influenced the choice of the strategy. Between the subjects of the 4^{th} grade that choice is independent of a direct teaching and the children choose the strategy in which they feel more confidence. (the additive strategy). The keyword problem pose a particular question. The strategy used by the children is connected with the teaching of the meaning of the keyword and so is not a surprise, that the choice of the multiplicative strategy increases in the 3^{th} and 4^{st} grades.

5. We have also notice that there are a very strong relationship between the correct mental representation and a correct answer in the exchange rate problem, keyword problem and multiple groups problem. The choice of the strategy is not related with the construction of the mental representation. Indeed, in all this word problems, and in all grades, we can find children who have construct a correct mental representation and choose the counting, the additive or the multiplicative strategy, and also subjects who fail in the construction of mental representation and choose the same strategies. In the combination problem all the subjects use the counting strategy and, here, it is obvious that the correct solution of the problem is more dependent on the way the strategy is used than on the construction of the mental

representation. Afterall, from a pedagogical point of view, it means that the teachers have to pay so many attention to the construction of the mental representation of a word problem as they normally pay to the construction of the strategies. We think that the use of concret material is a very useful way to help the students, particulary in Primary School, to be aware of the importance of this construction.

REFERENCES

DE CORTE, E. VERSCHAFFEL, L. - The effect of semantic structure on first grader's strategies for solving addition and subtraction word rpoblems. Journal of Research in Mathematics Education, 1987, 18, 363 - 381.

GREER, B. - Understanding of arithmetical operations as models of situations, in J. Sloboda and D. Rogers (Eds.), Cognitive processes in mathematics. Oxford, Clarendon Press, 1987.

RILEY, M., GREENO, J., HELLER, J. - Development of children's problem solving ability in arithemetic, in H. Ginsburg (Ed.), The development of mathematical thinking. New York, Academic Press, 1983.

SIEGLER, R. - Strategy choice procedures and the development of multiplication skill. Journal of Experimental Psychology, 1988, 117, 258-275.

RECONSTRUCTION OF MATHEMATICS EDUCATION: TEACHERS' PERCEPTIONS OF AND ATTITUDES TO CHANGE

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In Victoria, Australia, three quarters of the curriculum in all Year 11 and 12 mathematics subjects for has been turned over to individual projects, investigations and analysis tasks. External assessment has enforced radical change in mathematics education, with only one quarter of mathematics time now being spent on the teaching of facts and skills. A questionnaire was circulated to forty teachers to ascertain their perceptions of the resultant changes in their roles and to gage their attitudes to one particular non-traditional assessment task, a project on fractals. The data suggests that assessment is a more powerful determinant of classroom interaction than we generally acknowledge.

Introduction

Change in mathematics teaching, claims Barnes (1988), results from two main influences:

> ... forces from outside mathematics, real world problems for which a mathematical solution has been sought; and forces from within mathematics, such as the desire to unify and simplify, the search for extensions, generalisations and abstractions, and sheer intellectual curiosity and love of problem solving. (p. 25)

But as Dewey noted in 1916, social reproduction across a number of generations makes change difficult by reinforcing the concept of mathematics as an extensive record of knowledge, independent of its place as an outcome of inquiry and as a resource in further inquiry. Change forces thus create tensions for students and teachers, many of whom have gained the impressions that they are in classrooms to ensure the absorption of knowledge, rather than to play a part in its production. In today's classrooms, there is little sense conveyed that "because number systems have been created by human beings, they can be re-created or changed by human beings to meet the differing needs of more and more complex societies" (Frankenstein, 1988, pp. 74-75).

Like teachers of most western nations, Australian teachers feel like pawns in the mathematics education game. In the State of Victoria, for instance, curriculum development and control of innovation in mathematics education rests largely in the hands of the Ministry of Education's Curriculum and Assessment Board (VCAB) which outlines the expected mathematics curriculum and defines assessment tasks only for Year 11 and 12 students. This curriculum tends to drive the whole secondary curriculum and influences, to some extent, practices in primary mathematics education.

Superficial change

Although they have been reacting and adapting continually to changing pressures since the "new maths" of the early 1960s, Victorian teachers have not initiated major change in mathematics education and have rarely been involved in the development of its rationale. But forces from outside mathematics classrooms have been strong. Set theory, non-Euclidian geometries, group theory, probability theory, inequations, computer competencies: the list of

what counts as worthwhile mathematical knowledge in high schools has grown rapidly. There has also been a rapid transition from relatively separate disciplines ("Algebra", "Calculus", etc.), through "Business", "Pure" and "Applied" mathematics courses, to integrated mathematics courses; as well as a growing expectation that teachers and students will be competent in the use of a variety of modern technological aids.

One would expect, then, that mathematics classrooms of the eighties would be very different from those of the early sixties. However, teacher attitudes to change and their perceptions of the necessity for disturbing their comfortable patterns of curriculum content and social interactions in classrooms are powerful influences in the moderation of intended innovations. Despite the expectations of the Ministry of Education, tertiary institutions and employers, change in Victorian classrooms has been somewhat superficial. Teachers of mathematics education have been largely protected from imposition of initiatives undertaken in some other subject areas because of the discipline's traditional vantage ground and the widely-accepted cultural capital of its content. Any teachers involved in making curricular decisions have generally been drawn from the small percentage of scholars who have completed tertiary mathematics majors: for them there has been little necessity to interfere with proven recipes.

New mathematics did not seriously or lastingly challenge traditional notions of school mathematics. Aside from meeting powerful resistance through actively negotiated interpretation and selective implementation, it involved - in practice - very superficial changes in pedagogy. Textbook authors had no experience of alternative pedagogical and production styles, thus co-opting innovatory content to reinforce traditional teaching styles. Social interactions in classrooms also altered little. In 1975, Schoenheimer (pp. 7-8) noted that the new wave of new mathematics seemed to have

> crashed into a breakwater of granite-hard teacher conservatism. ... (with) little opportunity to explore, to discuss, to investigate, to reflect to formulate, to criticize, to develop, to create, to go wrong and to try a new attack - in a word, to learn mathematics by acting like mathematicians.

Exploratory activities were given little emphasis in assessment. Concepts and skills remained separated from contexts which give them meaning and teachers continued to use a traditional repertoire of procedures while transmitting the usual body of knowledge, despite changes in terminology and peripheral content. The weaknesses of the movement and the superficiality of its effects were also noted in a number of other countries during the 1970s (see, for instance, the reports of the Conference Board of Mathematical Sciences, 1975; Donovan, 1983; Thom, 1973; Stake and Easley, 1978; Weiss, 1978 and the studies reviewed by Fey, 1979a, 1979b).

Pedagogical styles have changed to the present day. In most upper-secondary (and hence lower-secondary and upper-primary) classrooms, discovery learning, problem-solving and problem-posing pedagogies and cooperative group work have all remained more rhetoric than reality: generalized aptitudes, competitive assessment and management of group learning and behaviour have continued to be foci of classroom interaction. This is understandable, given that teachers learn, though their own schooling experiences and training, to prepare and teach

lessons in which learning objectives, procedures and outcomes are largely pre-determined and in which co-ordination and control of both content and behaviour are primarily teachers' responsibilities. Students, too, are well trained in these expectations and are familiar with the consequent social roles. Active learning requires radically different patterns of communication, resources, classroom activity and assessment - but educators and pupils have little experience of these in the context of mathematics classrooms.

Assessment-controlled imposition of change

Thus it would seem that it is teachers who have constrained, to some extent, the introduction of real change in curriculum. Another powerful factor encouraging the retention of traditional curriculum has been the fact that the Year 12 matriculation examination has retained its implicit assumption that mathematical learning can be assessed adequately through three-hour, end-of-year, externally-set examinations.

In response to a call from politicians and employees for more emphasis on the development of skills appropriate to the changing needs of our society, and as part of a major overhaul of senior secondary education, The *Victorian Certificate of Education* (VCE) was introduced in 1990. Curriculum developers - including representatives of universities and other tertiary institutions, teachers, employers and parents - have taken this opportunity (through VCAB) to introduce radical changes to upper-secondary mathematics education courses. Common features of the nine new mathematics courses available in Years 11 and 12 include:

- Three quarters of the VCE mathematics is problem solving, project work and mathematical investigation. This balance is reflected across contact time, homework tasks and assessment. Facts and skills, traditionally the 'core' curriculum, are the focus for only one quarter of the courses. Thus much traditional content must be learned as it arises in the context of a variety of mathematical applications.
- Exams at the end of Year 12 have been replaced by four major externally assessed tasks - a project, an investigation, a facts-and-skills exam and an analysis (problem-solving) exam. While these external assessments are optional, they are necessary for university entrance. All students must also satisfactorily complete school-based requirements including attendance at classes as well as independent practice, application and investigation tasks.
- Mathematical content areas have remained integrated across all courses. Algebra, geometry, calculus, trigonometry and other areas are inextricably woven throughout the tasks. Course names (e.g. Space and Number, Reasoning and Data, Change and Approximation) reflect this conjunction, as do the topics set for student investigation (e.g. Waves, Low-energy light globes, Curves, Periodic time).
- Emphasis has been taken off the transmission of knowledge by teachers. Student investigations, independent construction and application of mathematics, researching of topics within communities, exploration of individual interests and extension of competencies, as well as other learner-centred forms of pedagogy, are not only encouraged but required.

As with the innovations of the sixties, the success of this program will depend very much upon teacher response and support. The changes, though, are imposed from outside of schools. They are assessment driven: the nature of the assessment tasks demand pedagogical innovation. The influence of the VCE is already flowing down through year levels.

This paper examines teachers' reflections on, and attitudes to, these radical changes: innovation which have re-defined not only content of texts, lessons and learning, but also styles of teaching and learning. The data suggests that assessment is a more powerful determinant of classroom interaction than we generally acknowledge.

Methodology

Foci of my research into VCE mathematics included the reactions of teachers to having students complete projects on topics related to mathematics in place of much of the traditional classroom and homework activity, and their attitudes to their new roles as facilitators, rather than controllers, of learning. This paper reports on their perceptions of one assessment task in particular: the *Space and Number* project on the theme *Fractals*, which seemed to stand out as a major challenge for most teachers and students.

Teachers were given only a few hours prior warning of the topic, then students had four weeks to conduct investigations and prepare their projects. Sixteen assessment criteria were presented with the topic. While assistance could be given with the mathematics arising from the investigation, teachers were not to pre-empt, control or take a major part in the investigation. Teachers were directed to allow about five hours of contact time for classwork on the topic, including systematic checking on student input and progress, but the majority of the project was to be completed out of school time. Most students took 30-40 hours to do their projects. Teachers then assessed projects and they were sent off for external verification or re-assessment. (Similar procedures were followed for a major mathematical investigation three months later.)

After teachers had assessed students' work, I contacted forty teachers with questionnaires and then conducted some follow-up interviews with teachers, students and parents. All of the questionnaires were returned, suggesting that teachers relished the opportunity of venting their reactions to being pushed into new ways of working. Only typical responses from teachers have been selected for this paper.

Results

While there were some very negative perceptions of both the project-style nature of the activity and the theme, these were balanced with some positive comments. Not all teachers resisted the new ways of working, and some became quite enthusiastic about the changes. Initial reactions, though, were unanimous: despite teachers having had extensive representation on decision-making committees, an hierarchy of power and a sense of imposition were perceived by the majority:

- It was close to April first! I waited for the real topic, but it didn't come. Then I got angry. What right did they have to introduce a new topic at Year 12 level?
- ... quite UNFAIR and obviously set by some VCAB TWIT who should be ...
  VCAB has no idea of the practical situations of classrooms.
- New topic, strange format and project-style work. Talk about guinea pigs! We were used to working through a rational, ordered and carefully selected series of topics. It didn't seem fair that they would suddenly thrust on us a project which threw everything into turmoil!

All VCE teachers had participated in in-service training days, but statements such as "responding to the changing nature of mathematics" had little meaning until they became reality. Having to cope with a new field of mathematics for an assessable task was not received kindly, particularly with so little time for preparation:

- Stunned, horror. Went green. Help!!! I went out and bought us all *Minties*.
- Panic. Fear of students' reactions. If it was something I hadn't heard of in 15 years of teaching, how would they feel? It has absolutely no relevance ... useless knowledge ... it wasn't in the text
- My boyfriend said it's not maths - and it's not. He didn't learn it at uni either.
- ... disbelief that this topic could be related to maths. There goes Year 12! Gloom and doom.

Expecting to retain traditional transmission-style of pedagogy, as opposed to being willing to explore a topic in a more equal relationship with students, teachers perceived their knowledge (both in content and teaching skill) as inadequate:

- My uncertainties were to do with my own adequacy of explanation and guidance.
- I knew NOTHING. But we sure learnt a lot in a hurry.
- I knew that after they got over the initial shock, they would ask me questions - and I wouldn't be able to answer them. But we usually managed to nut out something. Fun, really.
- The whole experience was threatening. I like to be in control of context. I felt I was learning too, but one step behind the workers.
- I had peripheral knowledge. But the students really taught me. Especially as I did the assessment. I didn't really understand what fractals are until I marked their (projects).

Teachers felt stripped of their normal roles of setting learning objectives, transmitting the necessary knowledge and skills, then assessing student performance:

- Since I had no idea what the students would be interested in, doing, or where they would be up to, I couldn't plan lessons. I didn't like not being able to lead them.
- I couldn't work out exactly what I was meant to teach over the next few weeks.
- Some teachers drew upon any resources available to retain familiar "teaching" roles.
- I developed some LOGO programs to demonstrate formation of the snowflake curve.
- ... identified skills needed, assessed student abilities, grouped them for skills teaching.
- ... found some teaching materials for each area I thought they might need.
- Our staff searched our library and all the tertiary libraries.
- I needed to find a strategy whereby my students could utilize my programming skills.

- Discussion was all one way. Completely teacher-centred, as students had no other resources apart from what I provided. I was able to set an aim for them and we worked out timetables.
- I explained all the different types of projects I had in mind. We decided to start with ...

Students, too, demanded that teachers play their usual role of leaders in the education process - not surprising after twelve years of transmission pedagogy:

- When students cannot see what is ahead, they put pressure on us to provide guidance - perhaps more than should be given. I was under a lot of pressure to design their projects.
- (Students) just sat back and waited for me to give them directions. It took them more than a week to realise I wasn't going to take charge.

Investigation into a topic which demanded a range of mathematics skills took time away from the regular classroom activities and demanded teaching, as different skills were required, from several chapters of textbooks:

- We had to cut back on skills and problem-solving. Now I can see that is what they were doing anyway, but at the time I felt we were getting behind. (It) was very different from the topics that we normally would have been covering. We had to jump all over the text book. Logarithms, sequences and series, mensuration - I have never done so much teaching in so few hours. It got quite exciting, but at the time I was thinking about resigning. Looking back, I can see it was the best professional development activity I have done since my first year of teaching (15 years ago!). It made me ... evaluate the traditional outcomes.

Perceptions of the change process included shifts in patterns of communication and in educational settings. In this situation where both teachers and students were "active meaning makers" (Cobb, 1988), constructing common understandings as they interacted with mathematical discourses, new social relationships developed temporarily:

- The kids put in many more hours that VCAB required. They called around to my place at night and popped into the staff room all day long. I hung around the library when they had study hall. I couldn't even walk through the (yard) without being tackled.
- All the questions were different, so we had little to discuss as a group. We timetabled regular consulting sessions in the staff room. Whoever was there joined in with ideas and references. I've never known the staff to work like that before.
- (The students) spent a lot of time helping each other... we made a resource folder and arranged for free photocopying. It was surprising how quickly the kids found articles to share. Some of them wrote computer programs together and were able to use these in their own individual ways. Being all in the same boat, they were quite co-operative for a change.
- I had to spend more time talking to individual students. This was good for student/teacher relationships. Discussion was much more open. Classrooms are usually pretty quiet.
- Our classroom became more activity centred than teacher centred.
- One student knew quite a bit about the topic. She virtually took over - gave us some great ideas. She's very reserved, so her competence with a strange topic was quite a surprise.

Some teachers realised they had made incorrect assumptions about student competence, either in mathematics or in skills associated with writing reports:

- Communication of the findings and report writing was a worry. I teach Maths, not English.
- They actually looked back through their textbooks to area and volume. I thought they were confident in those topics, but I soon found they couldn't apply them. It was hopeless. I was running around teaching Year 9 work. Then when I had taught Maths I had to teach English!
- I was just amazed at a few of them These were business maths students - teaching themselves from books, designing computer programs, using maths I never dreamt they could cope with. One girl said "Just show me which buttons to press." But before I knew it, she was explaining to her friend what logarithms were. She explained it much better than I could.
- Students' interests and needs could no longer be assumed to be the same across a class.

Discussion

Teachers' attitudes to the new certificate course have been affected by their perception of it being imposed "from above". While teachers were supportive of the principles of the VCE, such as making mathematics more relevant, student-centred and research-like, VCE mathematics has upset traditional patterns of discourse and knowledge production. It has also opened the way for study of new mathematical fields: a move which threatens the authority that greater content knowledge and traditional transmission procedures usually afford teachers. Requiring students to do mathematics as an activity rather than as a study of collection of propositions (Wittgenstein, 1956) engaged teachers in a style of pedagogy they have not been trained to cope with, and have rarely experienced in schools. While teachers recognise that relatively passive reception of content, pre-structured curricula and busy seatwork prevent institutionalised mathematics learning from being student-centred, changed practices have shaken common, comfortable constructions of classroom life.

However, the innovatory courses mathematics have created conditions for different styles of interaction in classrooms - and the extension of the learning arena past the school. It has demanded of students a lot more than do traditional mathematics classes, which are inclined to involve little more than a teacher-directed activity and the application of mathematical skills to the investigation of a relatively familiar topics and problems. Assessment-driven innovation has already forced less emphasis on performance of algorithms and more on those activities which foster mathematical creativity, initiative and investigation.

The setting of a project with an unfamiliar theme for one quarter of the externally-assessed component of a mathematics course put teachers and students at the heart of the process of developing new knowledge in the wider society. It demonstrated to them that mathematics is a constantly changing field, both socially created and defined and linked closely with other fields of knowledge. In exploring this new area together, students and teachers took a role in a small curriculum revolution - not only in the development of *what* was being taught, but also of *how* mathematics was being taught. This move to a situation where three quarters of the curriculum

is given to application, analysis and investigation has called into question the current linear curriculum practices which have their roots in a Newtonian paradigm,with its strong notions of predictability and control; and has suggested that both teachers and students can have a role to play in both learning and teaching. It has destroyed, to some extent, the traditional order of classrooms - but also encouraged the establishment of more positive ways of working.

As Clements (1989, p. 74)) points out ,"At the end of the 1980s, Victorian teachers of mathematics are being asked to change their ideas not only on what constitutes the purpose of school mathematics but also on how the subject should be taught." The setting of project work and mathematical investigation and analysis tasks, plus topics for which teachers and text book publishers can not merely transmit content and where the usual resources are not readily available, has proved the capacity of innovative assessment tasks to enforce new ways of organising mathematics classrooms. While these will at first be unfamiliar, they will possibly allow us to see teaching and learning in a new light. Throwing students in at the "deep end" of mathematical investigation, can encourage individual mathematical exploration and result in a range of disparate realised curricula.

While it is easy to believe that teachers control the curriculum, this study demonstrates that we can no longer doubt the power of assessment in determining pedagogical content and practice. But in the long run, as with the "new mathematics" movement of the sixties, teachers attitudes to the VCE and the extent to which they are prepared and enabled to change traditional practices will determine whether its innovatory features will become institutionalised.

References

Barnes, M. (1988). The power of calculus. *Education Links, 32* (1), 25-27.

Clements, M.A. (1989). *Mathematics for the minority: Some historical perspectives of school mathematics in Victoria.* Geelong: Deakin University.

Cobb, P. (1988). The tension between theories of learning and instruction in mathematics education. *Educational Psychologist, 23*(2), 87-103.

Conference Board of Mathematical Sciences, 1975; *Overview and analysis of school mathematics, grades K-12.* Washington DC : Author.

Dewey, J. (1916). *Democracy and Education.* New York: Macmillan.

Donovan, B.F. (1983). *Power and curriculum in implementation: A case study of an innovatory mathematics program..* Unpublished doctoral dissertation, University of Wisconsin - Madison.

Fey, J. (1979a). Mathematics teaching today: Perspectives from three national surveys. The Mathematics Teacher, 72, 490-504.

Fey, J. (1979b). Mathematics teaching today: Perspectives from three national surveys for the elementary grades). The Arithmetic Teacher, 27, 10-14.

Frankenstein, M. (1988). *A different third R: Radical math* (Vol. 1) Manuscript.

Marks, G. & Mousley, J. (in press). Mathematics education and genre: Dare we make the process writing mistake again. *Language and Education.*

Schoenheimer, H.P. (1975). Mathematics could be vital. In D.Williams (Ed.). *Perspective in teaching mathematics* (pp. 2-13). Melbourne: Mathematical Association of Victoria.

Stake, R. & Easley, J. (Eds.). (1978). *Case studies in science education.* Urbana: University of Illinois.

Thom, R. (1973). Modern mathematics: Does it exist? In A.G. Howson (Ed.). *Developments in mathematics education. Proceedings of the Second International Congress on Mathematical Education. (pp. 194-213).* Cambridge: Cambridge University Press.

Wittgenstein, L. (1956). *Remarks on the foundations of mathematics.* Oxford: Blackwell.

Weiss, I. (1978). *Report of the 1977 national survey of science, mathematics and social studies education..* Research Triangle Park, NC: Research Triangle Institute.

The Falsifiability Criterion and Refutation by Mathematical Induction

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Abstract

Popper's Falsifiability Criterion is matched with a learning analogue, to form the theoretical issue of testing the fragility of human knowledge. This is augmented by evidence from a naturalistic study of student-teachers' ability to resolve a conflict in mathematical induction. Coping with this task brought about students' reflections on the issue of refutation by mathematical induction, which they rarely had been considered.

The Falsifiability Criterion

Karl Popper's principal contribution to the philosophy of science rests on his rejection of the traditional inductive method in the empirical sciences. According to this traditional view, a scientific hypothesis may be tested and verified by obtaining the repeated outcome of substantiating observations. However, only an infinite number of such confirming results could prove the theory correct. Popper argued instead, that scientific hypotheses are deductively validated by what he called *the falsifiability criterion* (Popper 1961, Chapter iv). Under his method, scientists should persistently seek to discover an observed exception to their postulated rules. The absence of contradictory evidence, thereby becomes essential to the survival of the theory (Evans 1989, pp.42-44).

The ultimate goal of the process of learning in general, and of mathematics learning in particular, is to develop one's understanding to such a degree, that it is robust enough to withstand any attempt to create a cognitive conflict. In learning, in forming conceptual frameworks, the resolution of contradictory ideas

in a person's mind, thereby becomes essential to the survival of a learned theory, and to the process of crystalization of a cognitive construct. By analogy to Popper's falsifiability as a scientific characteristic of a theory, this paper suggests falsifiability as a characteristic of a cognitive construct.

The theory of learning I rely on stems from Piagetian theory of concept-formation through cognitive-conflicts. We internalize mathematical knowledge by doing mathematics. Understanding of mathematics is an enduring gradual process, of course. As Schoenfeld, Smith and Arcavi (in press) pointed out, the process of learning and understanding of mathematics is gradual, it is made out of a limited context, and it is also retrogressive. It is the latter that the empirical work I describe below was addressed to.

As one constructs ones own knowledge structures gradually, and within a limited context, gaps and loopholes are inevitable. No matter how wonderful the curriculum, nor how remarkable the teacher is, misconceptions, inconsistencies, confusions and fuzziness are bound to occur. Cognitive conflicts are therefore unavoidable.

My personal metaphorical view of the process of human-beings' knowledge accumulation, is borrowed from the well known mathematical problems of packing and stacking of spheres, and polyhedra. The existence of "air-bubbles" in a person's knowledge structure, makes this knowledge "fragile", easily falsifiable. As we do more mathematics the context gets wider, we encounter the gaps in our knowledge, and consequently our knowledge is refined. As a result it gets less fragile, namely less susceptible to unresolvable conflicts. Following Steiner (private communication 1989) I am using the term "knowledge fragility" to describe the state of our knowledge in the intermediate stage of the development, where our understanding is yet to be improved, and it is still in a shaky position, risky of being broken by some insufficiently careful move. The activities I designed for prospective-teachers, were aimed at accelerating the process of crystalization of their knowledge of particular mathematics notions, such as Mathematical Induction, by putting them in a problem-solving situations which will make them confront their present knowledge, and examine it carefully. Knowledge fragility is supposed to reduce, in the process. The underlying philosophy of this approach is that the more successfully their knowledge stands an attempt to inject contradictions into it, the more solid it gets. This approach is influenced by Karl Popper's falsifiability criterion.

Proof by Mathematical Induction

Giuseppe Peano (1858-1932) is usually credited for setting a set of postulates for the natural numbers, N. Peano's fifth postulates states that:*A subset of N which contains 1, and which contains $S(n)$ whenever it contains n, must equal N*. This is known as The Principle of Mathematical Induction (PMI). It should be noted that this postulate is concerned with subsets of the set of natural numbers, and *not* with propositions about the natural numbers. However, it can be applied to proofs of propositions of the form: "For all n, $P(n)$", where $P(n)$ is an open sentence, stating something about the natural number variable, n.

To apply the PMI to proofs of propositions, one needs to address oneself to the truth set of the open sentence $P(n)$, and show two things: (i) that this set contains 1, and (ii) that this set is inductive, namely that it contains with every natural number, its successor as well. This is logically equivalent to showing (i) that $P(1)$ is true, and (ii) that for *any* particular natural number k, the truth of $P(k)$, implies the truth of $P(k+1)$.

To complete the proof, having the two steps established, one needs to apply the basic rule of inference known as *Modus-Ponens*, to the Principle of Mathematical Induction, and deduce that the truth set of $P(n)$ is the whole set of natural numbers, or equivalently, that $P(n)$ is true for all n.

The Place of Mathematical Induction in the Curriculum

Many scholars have called attention to various aspects of learning and teaching of mathematical induction. (E.g. Avital and Hansen (1976), Dubinsky (1986), (1990), Ernest (1984), Fischbein and Engel (1989), Henkin (1960), (1961), Hanna (1989), Lowenthal and Eisenberg (in press), Movshovitz-Hadar (accepted 1991), Vinner (1976)). One challenged the explanatory potential of proofs by mathematical induction, others criticized the level of rigour attained. Still another questioned the contribution of the axiomatic presentation to students' deductive ability. However, there seems to be a general consensus that mathematical induction should be introduced in high school (NCTM 1989, p. 143). High school teachers, then, ought to be prepared to teach this topic. Because, as shown by Zaslavsky (1989) students' misconceptions can often be traced to their

teachers', teachers' and teacher-students' own understanding of this method of proof is very important.

The General Framework of the Present Study

The work described in this paper was carried out within the naturalistic framework of a mandatory problem-solving course, which is a part of the program leading towards a B.Sc.Ed. in mathematics education. This course serves two major goals: (i) to refresh students' high school mathematics, and reshape it where necessary, and (ii) to bridge between the college level mathematics courses, and the educational courses, particularly the methods course.

Conflict resolution activity proved itself instrumental in reaching these broad goals, in three ways: (1). Being put in a cognitive conflict helps students become aware of the limitations of their present understanding of certain concepts. (2). Resolving the conflict helps them develop a better understanding of these concepts. (3). Reflecting on both, helps them recognize children's learning processes and sympathize with their difficulties in the cognitive and in the affective domains. (For more details, see Movshoviz-Hadar and Hadass 1990, 1991).

In the Spring of 1990 mathematical induction was at the focus of an unforeseen large number of class meetings. Twenty four student-teachers participated. All had used Mathematical Induction to prove conjectures in high school.

Students' work

Students were assigned the following task for an individual consideration: Below are two contradicting responses given by two students to the same question. Whose answer, if any, is right? Where did the other one go

wrong? The question is: Is the following statement true or false

$$\frac{1 \times 3 \times 5 \times \ldots \times (2n-1)}{2 \times 4 \times 6 \times \ldots \times 2n} \le \frac{1}{\sqrt{2n}} \ ???$$

Student A

(i). For n=1 left side = 1/2, right side = $1/\sqrt{2}$.
Now, $2 > \sqrt{2} \Rightarrow 1/2 < 1/\sqrt{2}$
Hence, the inequality holds.

(ii). Assume it holds for n=k, i.e.,

$$\frac{1 \times 3 \times 5 \times \ldots \times (2k-1)}{2 \times 4 \times 6 \times \ldots \times 2k} \le \frac{1}{\sqrt{2k}}$$

We want to show that it holds for n = k+1, namely, that

$$\frac{1 \times 3 \times 5 \times \ldots \times (2k-1)(2k+1)}{2 \times 4 \times 6 \times \ldots \times 2k \times 2(k+1)} \le \frac{1}{\sqrt{2(k+1)}}.$$

By our assumption:

$$\frac{1 \times 3 \times 5 \times \ldots \times (2k-1)(2k+1)}{2 \times 4 \times 6 \times \ldots \times 2k \times 2(k+1)} \le$$

$$\le \frac{1}{\sqrt{2k}} \times \frac{2k+1}{2(k+1)} =$$

$$= \frac{[\sqrt{2k+1}\,]^2}{\sqrt{2k(2k+2)} \cdot \sqrt{2(k+1)}} =$$

$$= \frac{\sqrt{4k^2+4k+1}}{\sqrt{4k^2+4k} \cdot \sqrt{2(k+1)}} > \frac{1}{\sqrt{2(k+1)}}$$

This contradicts the statement and therefore it is false.

Student B

To prove the above it is sufficient to show that:

$$\frac{1 \times 3 \times 5 \times \ldots \times (2n-1)}{2 \times 4 \times 6 \times \ldots \times 2n} \le \frac{1}{\sqrt{2n+1}}.$$

(i). For n=1 left side = 1/2
right side = $1/\sqrt{3}$
$1/2 = 1/\sqrt{4} < 1/\sqrt{3}$
$\Rightarrow$ The inequality holds.

(ii). Assume it holds for n = k, that is:

$$\frac{1 \times 3 \times 5 \times \ldots \times (2k-1)}{2 \times 4 \times 6 \times \ldots \times 2k} \le \frac{1}{\sqrt{2k+1}}.$$

We want to show that it holds for n = k+1, namely, that:

$$\frac{1 \times 3 \times 5 \times \ldots \times (2k-1)(2k+1)}{2 \times 4 \times 6 \times \ldots \times 2k \times 2(k+1)} \le \frac{1}{\sqrt{2k+3}}$$

By our assumption:

$$\frac{1 \times 3 \times 5 \times \ldots \times (2k-1)(2k+1)}{2 \times 4 \times 6 \times \ldots \times 2k \times 2(k+1)} \le$$

$$\frac{1}{\sqrt{2k+1}} \times \frac{2k+1}{2(k+1)} = \frac{\sqrt{2k+1}}{[\sqrt{2(k+1)}\,]^2}$$

$$= \frac{\sqrt{(2k+1)(2k+3)}}{\sqrt{4(k+1)^2(2k+3)}} =$$

$$= \frac{\sqrt{4k^2+8k+3}}{\sqrt{4k^2+8k+4} \cdot \sqrt{2k+3}} \le \frac{1}{\sqrt{2k+3}}.$$

By induction it therefore follows that the statement is true for all n.

Only two of the 24 students got confused. These students wondered how come Student B's stronger assertion was provable, while the original assertion was not. To twenty two of the 24 students, it was apparently clear that Student A was wrong. However, pinpointing *where exactly* the error in A's work was, that was far from trivial. (More details - in the conference).

On top of the error detecting activity, this task provoked a very unusual examination of Mathematical Induction as a method of refutation. Reflections on this matter included :

- "The fact that you cannot prove a claim by MI does not mean that it cannot be proved using another method".
- "Student A used MI to refute and reject a claim, but I am not sure this can be done, at least, I have never seen it used this way ".
- "By mathematical induction one cannot refute a universal conjecture unless it is universally wrong, but then, where did it come from to start with? Its negation would make a more reasonable conjecture to try and prove by MI, right? "

Clearly, the students had to do some more thinking on their own, in order to clarify for themselves the power and the limitations of proof by MI. But, at least, they became aware they should. The last section is a brief summary of the lesson we were trying to draw following this experience. (For a wider scope, see Movshovitz-Hadar (submitted 1991).)

Discussion - Refutation by MI

A common error in conditional reasoning is to conclude from "If H, then C" that since H is false, C must be false too. (H, C, designate two independent sentences, the *hypothesis* and the *conclusion* of the conditional, respectively). This reasoning is fallacious. H might be false while C is true. The only thing which can't happen is H being true while C is false.

Now, what if for some k we find $P(k)$ to be false? Clearly, $P(k)$ is a counter example which refutes the assertion: For all n, $P(n)$. However, the conditional $P(k) \to P(k+1)$ can still be true for all k, while $P(n)$ isn't for all n. Moreover, in such a case, since $P(k)$ is not true, $P(k-1)$ must be false too, and this implies $P(k-2)$ is also false, etc., etc., down to $P(1)$. In other words, (i) cannot hold under these

circumstances, and surely, the PMI cannot be applied. Nevertheless, if there is a natural number t, greater than k for which the statement $P(t)$ turns to be true, then $P(n)$ still might be true for all natural numbers from this value up to infinity. (For an example, try to prove that the following sequence consists of either 1 or 2 for all its elements except possibly a finite number in the beginning: $a_1 = z$, where z is an integer; $a_{n+1} = a_n + 1$ if a_n is odd, and $a_{n+1} = a_n/2$ if a_n is even).

What if the validation step "does not work"? Namely, what if one does not succeed in proving either (i) or (ii) or both? The only valid conclusion in such a case is that the PMI cannot be applied to deduce $P(n)$. $P(n)$ may still be true, but to establish it one needs to look for a method of proof other than MI. Sometimes, we do get a result which refutes $P(n)$ *for all* n. Such a case occurs, for example, if $P(k)$ does imply the negation of $P(k+1)$, i.e. *not*-$P(k+1)$. In such a case it *is* valid to conclude that $P(n)$ is not universally true. (For an example, examine by MI the assertion: *For all* n, $5^n + 1$ *is divisible by 3).*

Concluding Remark

If mathematics-education is perceived as a meta-mathematical endevour, this paper intended to demonstrate the following meta-mathematical claim: Mathematical Induction is a complex domain for learning. Dealing with a knowledge-fragility testing task, has the potential of bringing about a deeper insight into previously acquired understanding of this complex mathematical domain.

References

Avital, S., and Hansen, R.t., (1976): Mathematical Induction in the Classroom, *Educational Studis in Mathematics*, 7, pp. 399-411.

Dubinsky, E., (1986): Teaching Mathematical Induction I, *Journal of Mathematical Behaviour*, Vol. 5, No. 3, pp. 305-317.

Dubinsky E., (1990): Teaching Mathematical Induction II, *Journal of Mathematical Behaviour*, Vol. 8, No. 3,pp. 285-304.

Ernest, P. (1984): Mathematical Induction: A Pedagogical Discussion, *Educational Studies in Mathematics*, Vol. 15, pp. 173-179.

Evans, J.B.T.,(1989): *Bias In Human Reasoning - Causes and Consequences*, Lawrence Erlbaum, Hilsdale, USA.

Fischbein, E., and Engel, I., (1989): Psychological Difficulties in Understanding the Principle of Mathematical Induction, in G. Vergenaud et als. (eds.): *Proceedings of the 13-th International Conference for the Psychology of Mathematics Education*, Paris, France, Vol. 1, pp. 276-282.

Hanna G., (1989): Proofs that Prove and Proofs that Explain, in Vergenaud et als. (eds.): *Proceedings of the 13-th International Conference for the Psychology of Mathematics Education*, Paris, France, Vol. 2, pp. 45-51.

Henkin, L., (1960): On Mathematical Induction, *American Mathematical Monthly*; Vol. 67, No. 4, pp. 323-338.

Henkin, L., (1961): Mathematical Induction, MAA film manual no. 1, *The Mathematical Association of America*, Printed by Cushing-Malloy, Inc., Ann Arbor, MI.

Lowenthal, F., and Eisenberg, T., (In press) Mathematical Induction in School: An Illusion of Rigour" *School Science and Mathematics*.

Movshovitz-Hadar, N. and Hadass R., (1990): Preservice Education of Math Teachers Using Paradoxes, *Educational Studies in Mathematics*, Vol. 21, pp. 265-287.

Movshovitz-Hadar, N., and Hadass R., (1991): More About Mathematical Paradoxes, *Teaching and Teacher Education*, Vol. 70, No. 1, pp. 79-92 .

Movshovitz-Hadar ,N., (accepted, 1991): Mathematical Induction - A Focus on the Conceptual Framework, accepted for publication pending revision, *School Science and Mathematics*.

Movshovitz-Hadar ,N., (submitted, 1991): The False Coin Problem, Mathematical Induction and Knowledge Fragility. Submitted for publication.

NCTM (1989): *Curriculum and Evaluation Standards for School Mathematics*., Prepare by the working group of the commission on standards for school mathematics of the National Council of Teachers of Mathematics, Reston, VA, USA.

Popper Karl R. (1961):*The Logic of Scientific Discovery*, Science Editions, Inc., New York. (Author's English translation of *Logic der Forschung*, originally published in Vienna, 1934).

Schoenfeld, A.H;. Smith, J.P.; Arcavi, A.(in press): Learning - The microgenetic analysis of one student's evolving understanding of a complex subject matter domain. To appear in R. Glaser (Ed.) *Advances in Instructional Psychology (Vol. 4)*, Hillsdale, N.J.,: Erlbaum.

Vinner Shlomo (1976): Implicit Axioms, ω-Rule and the Axiom of Induction in High School Mathematics, *American Mathematical Monthly*, Vol. 83, pp. 561-566.

Zaslavsky O. (1989): The Development of a Concept: A Trace From the Teacher's knowledge to a Student's Knowledge. Paper Presented at the Annual Meeting of the AERA, San Francisco, CA.

Young Children's Division Strategies

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This paper reports some of the effects of a classroom teaching experiment on young students' understanding of division and their ability to solve division-type problems. The experimental curriculum is built around students' construction of their own conceptually-based algorithms as a problem-solving activity, supported by a classroom atmosphere of discussion and negotiation. The meaning of division is developed through solving a mixture of partitive and quotitive division-type word problems, and students' algorithms become more sophisticated through a process of progressive schematization. Results indicate the use of powerful and correct theorems-in-action, and clear signs of meta-cognition in the form of a top-down approach to problem solving.

Introduction

Researchers generally agree that young children enter school with a wide repertoire of informal mathematical problem-solving strategies that reflect and are based partly on their understanding of the problem situation and partly on their existing concepts (Olivier, Murray & Human, 1990; Carpenter & Moser, 1982). Yet many children have severe difficulties with school mathematics, especially with the choice of operations for given word problems, and with the execution of taught algorithms. Instead of ignoring or even actively suppressing children's informal knowledge, and imposing formal arithmetic on children, instruction should recognize, encourage and build on the base of children's informal knowledge. Steffe and Cobb (1988) state: "In those cases where adult teaching is in harmony with the child's methods, the generative power of the child is extremely exciting and is unchartered (sic)" (p. 26).

Our research group is engaged in an ongoing research and development project on the mathematics curriculum in the first three grades of school, trying to build on children's informal knowledge and studying and facilitating the development of their conceptual and procedural knowledge (Murray & Olivier, 1989; Olivier, Murray & Human, 1990). This paper traces some of the effects of the experimental curriculum on young children's understanding of division-type word problems and the ways their solution strategies for these problems develop in a learning environment "in harmony with the child's methods" (Steffe & Cobb, 1988:26).

The Experimental Curriculum

The experimental curriculum and its research base have been outlined elsewhere (Olivier, Murray & Human, 1990), but its main characteristics are summarized briefly.

Our baseline study indicated that the majority of children invent powerful non-standard algorithms alongside school-taught standard algorithms; that they prefer to use their own algorithms when allowed to (or even when forbidden to!); and that their success rate when using their own algorithms is significantly higher than the success rate of children who use the standard algorithms or when they themselves use standard algorithms. This, coupled with a constructivist perspective on learning and the availability of calculators which necessarily leads to a re-evaluation of objectives for computation, has led us to formulate a teaching approach with the following main features:

- Strong emphasis on number concept development by helping children to construct increasingly sophisticated concepts of different units, including ten, and to build these concepts on children's counting-based meanings by encouraging increasingly abstract counting strategies and child-generated computational strategies. There are no number barriers, i.e. a particular teacher and her students can operate in any number range within the students' conceptual development.
- The development of the meanings of operations and solution strategies through true problem solving, i.e. meanings and strategies are *not taught*, but the teacher poses a word problem to a group of students and expects them to solve it in whatever way suits them individually. This is followed by a general discussion and comparison of methods used. The teacher does not suggest a method, and mistakes are identified and corrected by the group.
- A mixture of types of word problems are posed from the very beginning of grade 1, and are not classified as "addition" or "division" problems, since students select those operations that suit their strategies. For example, we believe that presenting students with both partitive and quotitive division-type word problems and requiring them to construct their own solution methods in response to the structure of each particular problem will firstly prevent discontinuities between the student's procedures and his concepts (Steffe & Cobb, 1988), and secondly enable him to construct an integrated meaning of division which makes possible eventual problem transformation. The idea of progressive schematization (Treffers, 1987) is implemented: The teacher starts with a general problem which the students solve by means of crude methods, and then creates a series of situations which will encourage students to refine their methods.

Mathematical notation is only introduced when students have trouble in documenting their solutions logically.

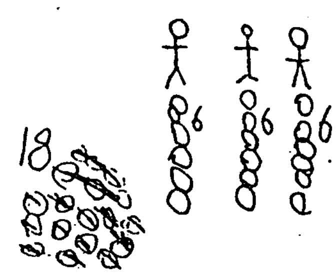
By the end of 1990 approximately 40 primary schools had already become part of the project, the number increasing to 170 in 1991. The data reported on here were gathered informally during weekly classroom visits by members of the research team to schools which had been implementing the experimental approach between three and eighteen months. We have available the written (or drawn) solutions of approximately 500 students gathered during these visits, as well as the workbooks of all the second and third grade students of two schools collected at the end of the 1989 academic year. The strategies utilized by students in the project schools not only reflect growing maturity of the division concept, but also the development of number concept. We describe below some typical strategies for division-type problems in more or less an order of increasing sophistication. It will be noticed from the descriptions of the different strategies that, although the five classes of strategies identified by Kouba (1989) are present, significantly more sophisticated and powerful strategies are elicited for computations involving larger numbers.

Different Strategies

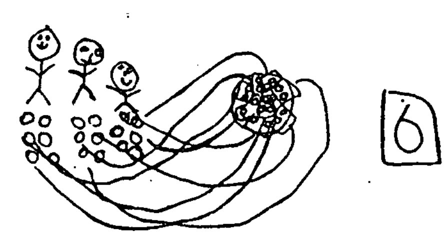
Direct representation

Although informal writing materials as well as counters are always available, it seems that students seldom use counters to model a problem. Rather, the problem context is drawn in greater or lesser detail, and then solved by further drawing in the actions needed. For example, Leana (grade 1) divides 18 cookies among three children one at a time, and Conrad (also grade 1) two at a time:

Leana



Conrad

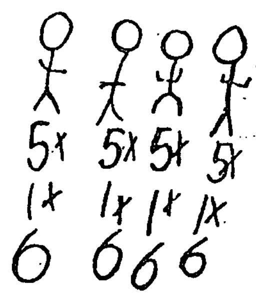


Numerical accelerated dealing

The student draws an iconic representation of the objects of the divisor (these may only be dots or they may be detailed drawings), but the number of objects dealt out in every round

are written below each icon as a *numeral*, for example Yolande (grade 1), who divides 27 balloons among four children.

Yolande



Although some children choose the number of objects to be dealt out per round according to the size of the iterable units they are able to cope with in that context (Steffe & Cobb, 1988), this accelerated dealing out strategy is optimized by sound estimation. There are two estimation-based strategies in this particular context: The first is a repeated-estimation strategy (trial-and-error). For example, to share 70 cookies among five children, a first estimate of ten turns out to be too low, a second estimate of 15 is too high but almost there, and the third estimate of 14 is just right. The second estimation strategy may be called an "estimate-and-adjust" strategy, where the first convenient estimate is corrected not by a new estimate, but by dealing out the remainder if the estimate was too low. Moana (grade 2) does the following:

66 ÷ 3 = 22
20 20 20
2 2 2

This estimation dealing strategy is quickly formalized by writing it as subtraction, addition, or multiplication sentences (see the following sections). It also forms a conceptual basis for applying the distributive property as illustrated in the section on multiples, for example, 70 ÷ 5 is solved as 50 ÷ 5 + 20 ÷ 5.

Subtraction

Subtraction as a method for division can represent three different conceptualizations:

- estimation dealing out for partitive problems. Emmerentia (grade 3) divides 81 apples equally among three boxes as follows:

  80 − 20 − 20 − 20 → 20 − 6 − 6 − 6 → 2 + 1 → 3 − 1 − 1 − 1 → 0 81 ÷ 3 = 27

- subtracting the number of objects dealt out in each round to solve a partitive problem, like Estelle (grade 1), who divided 18 sweets among three children as follows, explaining that she was "getting rid of" three sweets during every round of dealing out:

  18 − 3 = 15 → 15 − 3 = 12 → 12 − 3 = 9 → 9 − 3 = 6 → 6 − 3 = 3 → 3 − 3 = 0

- solving a quotitively-interpreted problem by repeatedly subtracting the divisor, for example, Antoinette (grade 3) solving 350 ÷ 70 (350 children, 70 children per bus, how many buses?):

  350 − 70 → 280 − 70 → 210 − 70 → 140 − 140 → 0 350 ÷ 70 = 5

We note that Estelle's interpretation provides a bridge between a partitive and quotitive interpretation of division.

Students do not often use subtraction as a method, and if they do, it is quickly replaced by other methods. For example, Etian (grade 3) was observed using the following strategies on three consecutive days:

96 ÷ 16:	96 − 5 − 5 − 5 − 5 − 5 − 5 − 5 − 5 − 5 − 5 − 5 − 5 − 5 − 5 − 5 − 5 → 16 − 1 − 1 − 1 − 1 − 1 − 1 − 1 − 1 − 1 − 1 − 1 − 1 − 1 − 1 − 1 − 1 → 0 5 + 1 = 6
94 ÷ 13:	13 + 13 + 13 → 39 + 13 + 13 → 65 + 13 + 13 → 91 1 + 1 + 1 + 1 + 1 + 1 + 1 → 7 rem 3
83 ÷ 13:	13 + 13 → 26 + 13 → 39 × 2 → 78 1 + 1 + 1 + 1 + 1 + 1 → 6 rem 5

Double counting, addition and multiplication

Double counting occurs in two closely-related forms, e.g. to compute 27 ÷ 3, the student may write down 3 + 3 + 3 + ... until the running total reaches 27, and then count the number of threes he had written down, or he may mentally count in threes, saying the running total or writing it down, and keep track of the number of threes on his fingers (both quotitive interpretations).

Addition and multiplication can be used for both partitive and quotitive interpretations of division. Stephen (grade 2) divides 18 sweets among three children by repeated estimation:

4 + 4 = 8 ~~5 + 5~~ 18 = 6 + 6 + 6

Students progressively formalize such strategies, eventually expressing them as multiplication, as Etian above. A dealing out strategy can also terminate in multiplication, for example 473 ÷ 12 = 39 rem 5:

initial version	30 + 30 + 30 + 30 + 30 + 30 + 30 + 30 + 30 + 30 + 30 + 30 = 360 7 + 7 + 7 + 7 + 7 + 7 + 7 + 7 + 7 + 7 + 7 + 7 = 84 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 = 24
final version	30 × 12 = 360 5 × 12 = 60 2 × 12 = 24. 2 × 12 = 24

Henriëtte (grade 3) uses multiplication in the repeated-estimation strategy she employs to solve 278 ÷ 12:

6 × 25
120 + 120 → 240 + 60 = 300 too many.

6×23
$240 + 36 = 276$ 23 and 2 left over.

Decomposition of the dividend into multiples of the divisor

This method indicates the ability of the student to reconceptualize a number as the sum of multiples of iterable units, and is by far the most common strategy among the more experienced students. The strategy also includes a fair amount of estimation and the use of known number facts. For example

for $51 \div 3$: $30 \div 3 = 10$; $12 \div 3 = 4$; $9 \div 3 = 3$; $10 + 4 + 3 = 17$ (Gerhard, grade 2)

and for $70 \div 5$: $12 \times 5 = 60$; $2 \times 5 = 10$; $60 + 10$; $70 \div 5 = 14$ (Jean Pierre, grade 2)

An initial decomposition may prove of no value, and a second one is tried, as Botha (grade 3) found for $53 \div 3$:

$53 \rightarrow 50 + 3 \rightarrow 30 + 21 + 2$
$10 + 7 + 0 = 17$ rem 2

A partial implementation of the decomposition strategy leads to the following "bulldozer" method of modular arithmetic—Theo (grade 3) computes $54 \div 3$ as follows:

$54 \rightarrow 50 + 4$ $30 \div 3 \rightarrow 10$ $12 \div 3 \rightarrow 4$ $8 \div 3 \rightarrow 2$ rem 2
$4 \div 3 \rightarrow 1$ rem 1 $10 + 6 \rightarrow 16$ rem 2 + 1 rem 1 $\rightarrow$
17 rem 3 rem $3 \rightarrow 1$ $17 + 1 \rightarrow 18$ rem 0
$54 \div 3 = 18$

Discussion

Transformation

Our data indicate that, although solution strategies initially closely model the problem structure, students eventually construct an integrated meaning for division if they had been exposed to different word problem types, and they become able to select a method independent of the problem type, but which best suits the numbers involved. The more experienced students choose multiples or estimation or a combination of both, regardless of whether the problem was quotitive or partitive. These more advanced strategies indicate a top-down approach to problem solving in the sense that the student constructs an overview of the problem, then decides which basic structures that he already has available he can use in this instance (called "plans" in computer programming (Soloway, 1985)).

Transforming a number in order to apply a multiple as a known number fact is extremely common. Here follows a slightly more complex transformation: To compute $76 \div 4$ the following change and compensate method is frequently used:

$80 \div 4 = 20$ $4 \div 4 = 1$ $20 - 1 = 19$ $76 \div 4 = 19$

Division by four, accomplished by two successive halvings of the dividend, is common. Division by five by doubling the dividend and then dividing by ten is less common, as is Mario's (grade 3) method of dividing by 15:

$105 \div 5 \rightarrow 21 \div 3 \rightarrow 7$

Both methods indicate the use of quite advanced theorems-in-action (Vergnaud, 1988). We have found that children easily progress towards using these theorems-in-action to cope with larger numbers. We give two examples from a grade 2 class who had been asked to compute $4158 \div 11$:

Bernie: "I first did 300×11 in my head. That gives 3300. The I took 80×11 because I wanted it to be 880 but then I saw it was too much. So I decided on 78×11 which was right. So the answer is $300 + 78 = 378$"

Sheryl: "$300 \times 11 = 3300$
$50 \times 11 = 550$
Then I decided to use 25×11 because I have just now used 50×11 and I can halve 550. Then I added 3×11. So $300 + 50 + 25 + 3$ gives 378."

The Role of Discussion

A crucial aspect of the experimental approach is the role of discussion among students to promote reflection (compare, for example, Cobb, Yackel & Wood, 1988). This has two main functions, firstly the improvement of methods by reflecting on one's own and others' methods, and secondly the prevention of misconceptions taking root and the clarification of errors.

The following serves as an example of how quickly methods can be improved: A group of ten second-graders whose school had not yet joined the project were asked to share 27 sweets equally among three children. Eight students drew a direct representation and shared out the sweets underneath a drawing of a child's face, and two students drew three faces but dealt out five each during the first round and two each during the next two rounds. After they had been asked to explain their thinking to each other (which they did with great pleasure and enthusiasm), they were asked to divide 37 sweets among three children. This time, only one student drew a direct representation, sharing out one at a time, whereas the other nine students used either two rounds of five each or one round of ten each, followed by a round of two each.

Discussion is also an extremely powerful tool for self-correction. During a visit to a grade 3 classroom in a project school, it was observed that, although the students were very proficient in dividing by one-digit numbers, their teacher had never confronted them with two-digit divisors. A word problem (93 cookies, 13 per bag, how many bags?) was posed to

a group of five students. Four students constructed a method in analogy with the decomposition of numbers suitable for addition, e.g. 90 + 10 = 9; 90 + 3 = 30; 3 + 3 = 1; obtaining an answer of 40 remainder 3. Wimpie chose a strategy based on multiples of 13 and obtained an answer of 7 remainder 2.

During the subsequent discussion they realized that 40 was completely unacceptable. Lynda and Etienne estimated ten (not accepting Wimpie's strategy *or* answer), tested ten as a possible solution and only then tried seven. But the very next problem posed (78 + 15) was solved like this:

Lynda and Etienne:	78 + 15 = 5 rem 3
	10 × 6 = 60 5 × 6 = 30 (choosing 6 as a first estimate)
	10 × 5 = 50
	5 × 5 = 25
The others:	78 + 15 = 5 rem 3
	60 + 15 = 4 18 ÷ 15 = 1 rem 3
	4 + 1 = 5

It must be noted that these children solve division-type problems *before* they had been taught about division or the symbol for division: It is therefore not only a viable but also a successful teaching strategy to base teaching on word problems and children's own solution methods, in an environment of discussion and negotiation. These children do not experience the solution of division-type problems as more difficult than any other problem types, but find them interesting and stimulating.

REFERENCES

Carpenter, T.P. & Moser, J.M. (1982). The Development of Addition and Subtraction Problem-Solving Skills. In T.P. Carpenter, J.M. Moser & T.A. Romberg (Eds.), **Addition and Substraction: A Cognitive Perspective** (pp.9–24). Hillsdale, NJ: Lawrence Erlbaum Associates.

Cobb, P., Yackel, E. & Wood, T. (1988). Curriculum and Teacher Development: Psychological and Anthropological Perspectives. In E. Fennema, T.P. Carpenter & S.J. Lamon (Eds.), **Integrating Research on Teaching and Learning Mathematics** (pp. 92–130). Madison, Wisconsin: Wisconsin Center for Education Research, University of Wisconsin-Madison.

Kouba, V.L. (1989). Children's Solution Strategies for Equivalent Set Multiplication and Division Word Problems. **Journal for Research in Mathematics Education, 20** (2), 147–158.

Murray, H. & Olivier, A. (1989). A Model of Understanding Two-Digit Numeration and Computation. In G. Vergnaud, J. Rogalski & M. Artigue (Eds.), **Proceedings of the Thirteenth International Conference for the Psychology of Mathematics Education**, Vol. 3 (pp. 3–10). Paris, France.

Olivier, A., Murray, H. & Human, P. (1990). Building on Young Children's Informal Mathematical Knowledge. In G. Booker, P. Cobb & T.N. Mendicuti (Eds.), **Proceedings of the Fourteenth International Conference for the Psychology of Mathematics Education, Vol. 3** (pp. 297–304). México.

Soloway, E. (1985). Learning to program = learning to construct mechanisms and explanations. **Communications of the ACM, 29**, 850–858.

Steffe, L.P. & Cobb, P. (1988). Multiplicative and Divisional Schemes. **Focus on Learning Problems in Mathematics, 10** (3), 11–28.

Treffers, A. (1987). **Three Dimensions: A Model of Goal and Theory Description in Mathematics Instruction – The Wiskobas Project.** Boston: D. Reidel.

Vergnaud, G. (1988). Multiplicative Structures. In J. Hiebert & M. Behr (Eds.), **Number Concepts and Operations in the Middle Grades**, Vol. 2 (pp.141–161). Reston, Virginia: NCTM–Erlbaum.

"IT MAKES SENSE IF YOU THINK ABOUT HOW THE GRAPHS WORK. BUT IN REALITY..."

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This paper reports a study of how high school students interpret and assimilate experimental data to validate and construct mathematical models. Two teaching interviews were structured around a sequence of problems involving the translation between functions of position and velocity. A computer-based motion detector enabled the students to design and discuss experiments with motion. Our analysis focused on learning episodes through which the students modified their views and expectations. A description of critical aspects of these learning processes is elaborated.

There are two fundamental sources of evidence in math and science: mathematical proof and experimental data. In spite of the fact that the first is usually associated with mathematics and the second with science, both are intimately coupled in the genesis of knowledge. So far, there has been more research about the understanding of mathematical proof (Balacheff, 1990; Fishbein, 1982), than about the use of experimental data in the learning of mathematical concepts. This paper aims to add to the latter by discussing the dialectic between experimental data and a variety of mathematical models in students' learning.

Often teachers assume that an experimental demonstration, either by the teacher herself or embedded in a laboratory activity, is a convincing and engaging way to prompt students' adoption of new ideas. However, in a development parallel to the research on mathematical proof, there is a growing evidence that the students may assimilate or dismiss the experimental evidence in ways that are very different from the teachers' expectations (McDermott et al, 1987; Monk, 1990).

This study is part of a larger project funded by the National Science Foundation (Grant #MDR-8855644) involving the learning of calculus concepts by high school students. In the context of this project, called Measuring and Modeling, we designed a sequence of problems involving the translation between a function and its derivative, and we are studying students' learning in three different situations: motion, air flow, and computer-based discrete modeling. We will report in this paper results from four teaching interviews with students working with motion. The students experimented with the motion of a car attached to a motion detector to explore mathematical patterns in the translation between functions of position and velocity versus time.

Since all the experiments carried out by students involved the representation of motion in Cartesian graphs, issues surrounding graphical representation of motion are at the core of our inquiry. Difficulties and misconceptions manifested by students in producing, interpreting, and translating

position and velocity graphs have been studied by many researchers (Clement, 1985, Bell and Janvier, 1981, Hitch et al, 1983). Our research is distinguished by the combination of three important characteristics: 1) we posed problems related to the actual motion of a manipulable car (rather than situated in imaginary contexts), 2) we focused on learning episodes (beyond a taxonomy of students' difficulties), and 3) we used computer-based systems, called Microcomputer-Based Labs or MBL, to measure and generate graphs in real time, as opposed to measurements with rulers and clocks.

The use of MBL for the study of motion has been reported as a highly valuable tool for students' learning (Thornton and Sokoloff, 1990). The system that we use measures and displays curves of velocity and/or position, in real time, for any moving object aligned with the motion detector. The software allows the user to choose which functions to represent and to compare two different sets of data. The software encourages the student to focus on global shapes rather than a pointwise analysis of functions, but an option is available for the student to look at the numerical values of the measurements. The learning environment within which the student operates has three components: the graphical representation of position vs time, the graphical representation of velocity vs time and the situation of the actual car with the motion detector.

Students tended to make both syntactic and semantic connections within this environment. A syntactic translation is defined as a rule of correspondence between the two representations (e.g. the slope of a line in position vs time indicates the height of the corresponding horizontal line in velocity vs time). A semantic translation is one that is thought of in terms of a sequence of events with the motion of the car. A semantic translation is accomplished by considering a particular motion compatible with one of the functions and inferring the other function as it would occur in the same motion.

We worked with individual students in two sessions of 90 minutes through a sequence of 12 translation problems. All our students were in high school and had not taken courses in calculus. The problems were posed by drawings on a newsprint pad, asking students what they expected for one of the functions knowing the shape of the other. The students decided which experiments to carry out; they drove the car with their own hands, adding a kinesthetic perception of the motion. The discussion followed through the production/interpretation of both experimental graphs and graphs drawn by hand.

Through our analysis of the transcripts we strove to find what we call guiding ideas, namely, ideas that helped the student articulate specific predictions and for which the student expressed a certain commitment. Many guiding ideas corresponded to syntactical rules (e.g. straight lines in velocity correspond to

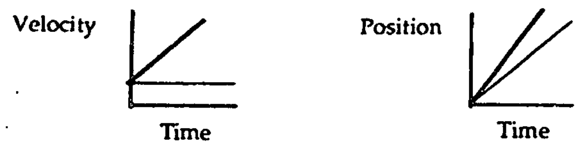
straight lines in position). Often a student constructed a set of guiding ideas which were not compatible from a formal point of view, seeming to emerge from different aspects of the situation without internal consistency. Most of the learning episodes that we have observed included experiences in which the contradictions between different guiding ideas became salient to the student. Given that condition of awareness it became more likely for the student to construct a semantic translation as a touchstone of her restructured knowledge.

The following two stories are excerpted from transcripts of students using the car and the graphical representation of its motion to explore the mathematical relationship between position and velocity. The students' struggles to find a consistent explanation for the data they confronted provides some insight into three aspects of the data/model dialectic: (1) How expectations about a model affect data interpretation; (2) The role of experimental situations and graphical models as evidence; (3) How a measurable, physical model (in this case, the car with motion detector) may connect the syntax and semantics of representations, leading to a more coherent mathematical model.

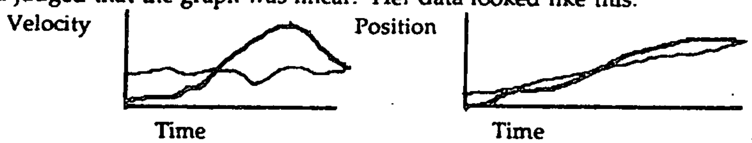
Nancy's session At the beginning of her session, Nancy investigated several velocity and position graphs in situations of constant positive velocity or constant positive acceleration. Through those experiences, she inferred the following guiding ideas (none of which is actually correct): (1) A straight line on the velocity graph corresponds to a straight line on the position graph and vice versa; (2) Position and velocity lines are parallel; either both are horizontal or neither is; (3) Constant velocity (a horizontal line on the velocity graph) corresponds to a diagonal (roughly 45°) line on the position graph, regardless of the value of velocity.

Besides being incorrect, these ideas are not consistent; ideas 2 and 3, for example, predict different position graphs for horizontal (constant) velocity. Nancy, perhaps sensing this inconsistency, often expressed a feeling of "not being sure;" on the other hand each of these ideas was robust in the sense that she used it to make predictions in situations she judged to be similar.

At the beginning of this episode, Nancy (correctly) predicted that for a constant velocity line, the position graph would be a diagonal (idea 3). To check her prediction, she used the car to produce a constant velocity graph and looked at the corresponding position graph. She felt that the sloping position graph supported her expectation. The interviewer then drew a position graph with a steeper line and asked her for the corresponding velocity graph. Nancy drew a straight line (idea 1), but slanted it so that it was parallel to the position line (idea 2):



Again, Nancy used the car and motion detector to test her prediction, creating a sloping velocity graph and looking at the derived position graph. In this case, the position graph was actually <u>curved</u>, but Nancy did not see the curve and instead judged that the graph was linear. Her data looked like this:



Nancy's interpretation of her data graph can be described as an interaction between the ambiguity of the graph and her expectations. Every graph is ambiguous to some degree; because of its scale, the curve on Nancy's graph could easily be seen as nearly linear. And because Nancy's task was to decide whether the graph she generated was consistent with her "parallel lines" theory, she was predisposed to see any curve that was not <u>clearly</u> curvilinear as linear. Thus, she found it natural to interpret the ambiguous curve in her graph as a line. In addition, she backed up her judgment with some practical experience with the motion detector: straight lines are sometimes difficult to generate, so it is reasonable to interpret slightly irregular "lines" in the data as matching straight lines in the graphical model.

Aware of this interpretive problem, the interviewer asked Nancy to compare two different constant velocities, hoping that she would recognize the incorrectness of both ideas 2 and 3 when she saw the diverging position lines. Consistent with idea 3, Nancy anticipated the same diagonal line on the position graph for both velocities. Her generated data, however, showed two lines that clearly diverged as they got further from the origin.

Here, Nancy confronted a clear contradiction between her hypothesis and the data. As before, she attempted an explanation in terms of experimental artifacts, but the separation between the lines was too extreme to be produced by inaccuracies in the motion detector system. She tried hard to claim that the lines were parallel: "just like those two lines are parallel [velocity lines], the distance lines are parallel?", but she recognized a few moments later that "...it seems like they [the distance lines] come, go further apart".

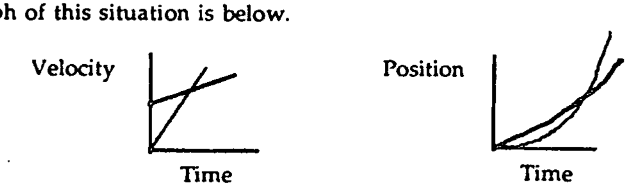
Nancy was visibly puzzled at the contradiction, but she could not make sense of the experimental result. The interviewer asked her "How do you imagine

the two cars moving according to this [the velocity graph]?" In response, Nancy simulated the motion of the two cars, starting from the same position at different speeds. She noticed that the two cars moved further apart and, reconsidering the distance graphs again, Nancy made sense of the two lines separating themselves. She moved her hands to imitate two cars proceeding at different velocities, moving further and further apart:

> "Well, 'cause if they start, if they are going by the same time, so like they started together, they're both going at a steady pace, so then the one that's covering ground faster, it's, it's just speeding ahead like that. So they'd be like, off like that" (holds her 2nd and 3rd finger in a V, mimicking the position graph.)

Nancy's breakthrough here demonstrates the power of the experimental apparatus to act as a bridge between two abstract entities: the velocity graph and the position graph. In the beginning of the session, the car functioned for her as a way to discover a relationship between a velocity and position graph, as Nancy reproduced one of the two graphs and compared the other with her prediction. Toward the end of her investigation, the cars functioned as a context for a thought experiment in which she constructed a coherent story linking the behavior of the cars with the appearance of the graphs. Thus, the car and motion detector provided a way for her to progress from the syntactic rules she had developed ("Horizontal velocity lines produce the same diagonal position lines.") to a semantic connection between the cars' positions and velocities ("the one that's covering ground faster, it's just speeding ahead").

<u>Steve's session</u> Steve's session demonstrates a similar role for the car and motion detector as a bridge between partial understandings of the mathematical model. Steve spent a long time in his second session dealing with the classic problem: If two cars with constant but different accelerations have the same velocity at time t_1, at what time do they meet (i.e. have the same position)? A graph of this situation is below.

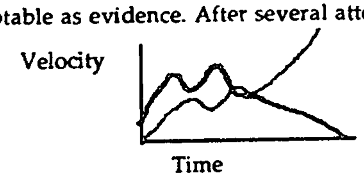


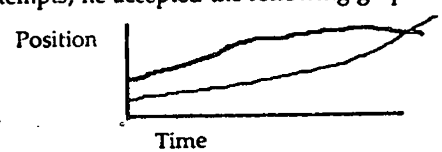
Steve's first response to the problem was the common incorrect notion that the cars are at the same position when they are going the same velocity. He moved the car to illustrate the following argument:

> That's where they would end up being side by side if they started at two separate places and ended up side by, they would end up side by side at that point where the lines intersect.

Steve's first response illustrates one aspect of the data/model dialectic. In order to investigate formal aspects of the model, Steve used real-world knowledge that the physical objects evoked. He generalized (incorrectly) from his experience of riding in cars to assert that if two cars were going the same velocity, they must be riding "side by side," since, in the real world, that is the simplest way to know that two cars are going at the same velocity.

To investigate his hypothesis, Steve went on to produce the two velocity lines. In the process, he had some difficulty producing a velocity graph representing a steadily increasing velocity. He tried several times; each time he had to decide when a data graph was close enough to the graphical model to be acceptable as evidence. After several attempts, he accepted the following graph:





Steve's decision is reminiscent of Nancy's; his expectations heavily influenced the way he judged the data. He accepted the graph above based on features that it shared with the target graph: the two lines started at different velocities, they met at a point, and when they met, the top line's y-value had increased less than the bottom one. An unbiased viewer, however, might have described the graph quite differently, since the first segment of the top line actually increased <u>more</u> than the bottom line, the opposite of what Steve was trying to achieve! Yet for Steve, the similarities were sufficient.

Looking at his graph of the car's movements shook Steve from his conviction that the cars would meet when they were going the same speed, but it left all possible meeting places open. From this point on, his focus changed to an extended experimentation with the process that produced his data graph. He made many mock runs with the car, trying to visualize how the two cars might look when they moved at different velocities. Yet, he did not sketch any graphs. It seems that at this point he was trying to make sense of his data graph by constructing a convincing story about the motion of the cars. However, rather than measuring the cars' motion and examining the resulting graphs, he operated more informally, moving the cars in different ways until their motion made sense to him. Here Steve, like Nancy, appeared to regard regularities he could observe in the motion of the cars as better explanations than characteristics of a graph.

While graphs are regarded as more concise and universal in their communicative power, both Steve and Nancy preferred a more story-like description of the patterns they saw in their data and used the cars as the protagonists of the story. For them, stories were compelling; graphs were formal and abstract.

After a while, Steve was able to articulate a convincing rationale for why the velocities would be equal <u>before</u> the distances were equal <u>in general</u>. His description was accompanied by many linguistic references and hand motions that recalled his experiences with the cars.

> "The reason is that they're starting at the same place; however, this one --I'll call this number one --is starting at a much faster speed so while this one is sort of (rrr noises indicating slow velocity) this one is (whroom noise indicating fast velocity). And it's accelerating at a steady pace of whatever, although it's accelerating a small amount while this one is accelerating a lot. So that when they're going equal speed, say this one would be this far behind this one, and then suddenly it's going to catch up a little later cause this one just isn't accelerating as fast as this one."

Both this explanation and other details of the video transcript indicate Steve's reliance on thinking in the world of the cars. Having a coherent picture of how the cars might move in the situation depicted by the graph was the glue that joined his partial understandings of the graphical models. The car environment provided a familiar context in which Steve could tell a story that explained not only the particular behavior represented by the problem graph, but the entire class of problems involving two cars with different constant accelerations. Working with the cars enabled him to understand the position and velocity graphs as two reflections of a semantically coherent situation.

<u>Conclusion</u> Nancy's and Steve's sessions have much in common, even though they struggled with problems of different complexities. In both cases, we saw their expectations influence their judgments about the relevance of a data graph to their hypothesis, as they interpreted the ambiguity of a graph "in their favor." We saw the car/motion detector system play a significant bridging role between syntactic and semantic understanding of motion for both of them , as it provided an experimental environment in which they could play out their hypotheses without the rigor of a graph. In the end, each of them understood the situation in terms of a story about the cars. This tendency seemed to reflect a belief that a story about the cars' behavior embodied inescapable truth - while the corresponding graphs were inherently more ambiguous and less compelling.

All three of the general points of this paper reflect the complexity of the data/model dialectic. Sometimes students find themselves in the uncomfortable position of believing something in one context that does not appear to be true in

...er; "solving" a problem often consists of negotiating between these two ......" Nancy, finding herself in the midst of such a dilemma trying to relate the graphical behavior with the actual motion of the cars, expressed it succinctly "It makes sense if you think about how the graphs work, but in reality

Balacheff, N. (1990) Beyond a psychological approach of the psychology of mathematics education. Plenary Address. *Fourteenth international PME conference.* Mexico.

Fishbein, E. (1982) Intuition and proof. *For the Learning of Mathematics*, 3(2), pp.9-18

Monk, G.S. (1990) *Students' understanding of a function given by a physical model.* Unpublished manuscript. University of Washington.

McDermott,L., Rosenquist,M., and vanZee,E. (1987) Student difficulties in connecting graphs and physics: Examples from kinematics. *American Journal of Physics*, 55(6), pp. 503-513.

Clement, J. (1985). Misconceptions in graphing. In L. Streefland(Ed.) *Proceedings of the ninth PME international conference.* (Vol. 1, pp. 369-375). Utrecht. The Netherlands: IGPME.

Bell, A. and Janvier, C. (1981) The interpretation of graphs representing situations. *For the learning of mathematics.* 2(1), pp. 34-42

Hitch,G.J., Beveridge, M.C., Avons, S.E., and Hickman, A.T. (1985) Reference domain on children's comprehension of coordinate graphs. in *Acquisition of Symbolic Skills*, D. Rogers and J.A.Sloboda, (Eds.) New York: Plenum Press.

Thornton, R.K., and Sokoloff, D.R. (1990) Learning motion concepts using real-time microcomputer-based laboratory tools. *American Journal of Physics*, 58(9), pp. 858-867.

TWO-STEP PROBLEMS - RESEARCH FINDINGS

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Introduction

Based on research of additive and multiplicative word problems, the present work deals with two-step arithmetic word problems and their structure. To facilitate communication for the purposes of this paper, the terms used will first be explained. In particular, what is meant by "component", "underlying structure" and "scheme". Let us start with an example of a one-step problem.

Problem 1: There are 13 boys and 16 girls in the group. How many children are there in the group?

In a properly formulated one-step problem, the text contains three components (propositions), two of which convey the numerical information (complete components) and one, the question component, which lacks the numerical information (incomplete component).

The three components in the above problem are:

(1) 13 boys in the group (complete).

(2) 16 girls in the group (complete).

(3) How many children are there in the group? (incomplete).

We speak of an additive relation rather than addition and subtraction operations because the identical situation (29 children in a group composed of 13 boys and 16 girls, in this case) can yield three different problems, depending on the incomplete component (whether we ask of the boys, the girls or the children). We claim that the incomplete component is crucial for understanding that Problem 1 calls for addition. If after the two complete components the incomplete component would be "How many more girls are there in the group than boys?" instead of "How many childen are there in the group?", the problem would become completely different. We call this 3-place relation contained in each one-step word problem the "underlying structure" of the text (or the "structure") which always consists of three components. Finally, the term scheme as used in this paper, means a combination of two or more structures. It will be further

clarified when we discuss two-step problems. The characteristics of the schemes for two-step problems are the target of our study. A two-step problem calls for two binary operations to solve a problem. For example,

Problem 2: There are a total of 35 flowers, distributed equally among 7 vases. In each vase there are 3 tulips and the rest are roses. How many roses are there in each vase?

The explicit components of this problem are:

1) A total of 35 flowers distributed equally among vases (complete component).
2) 7 vases (complete component).
3) 3 tulips in each vase (complete component).
4) How many roses are there per vase? (incomplete component).

Based on our analysis of one-step problems, we would expect to find in two-step problems Two underlying structures, and therefore six components. But so far we have identified in the text only three numbers and four descriptors. How is the rest of the information that enables solving the problem obtained? we claim that each two-step Problems contains an additional latent component which can be deduced from the given text even though not mentioned explicitly. In the above example the latent component is "How many flowers are there in each vase?". The latent component is shared by both structures and serves as the link between the two structures.

Categorization of Two-Step Problems

A. Categorization by Operation

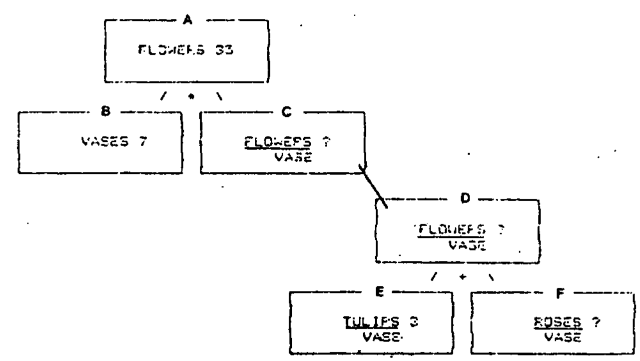
Pioneer work by Gray (1940) suggested categorization according to the operations to be performed in the problem. If we restrict ourselves to 2 binary operation problems, and consider all possible combinations of the four basic mathematical operations (addition, subtraction, multiplication and division) we arrive at 16 distinct problems. When non-commutative operations are involved, the order of the operations needs to be considered to distinguish, for example, between a:(b+c) and (b+c):a; or a:b+c. The above three phrases denote a more basic distinction, i.e.,

which operation will be performed first. Thus, taking into account the order of the operations, we arrive at 24 distinct problems.

B. Categorization by Schemes

Consistent with our textual analysis, the basic unit of our analysis should be a complete structure and not an operation. Thus, while Gray has four basic building blocks (+,-,x,:) for elementary two-step problems, we propose just two building blocks: the additive structure and the multiplicative structure.

Schemes for solving two-step problems have been widely used in Israel during the last fifteen years (1975) and served as motivation for Shalin's Dissertation (1985). To illustrate, let us first examine the scheme for Problem 2. Problem 2 consists of a connection between two structures (additive and multiplicative). The connection is formed so that the sum of the additive structure becomes one factor in the multiplicative structure. The Solution for Problem 2 will be calculated by A:B-E and the general form of the scheme for Problem 2 will be:



Let us now examine Problem 3, which has the same underlying structures and actually describes the same situation, but where designation of the complete, incomplete and latent components differs.

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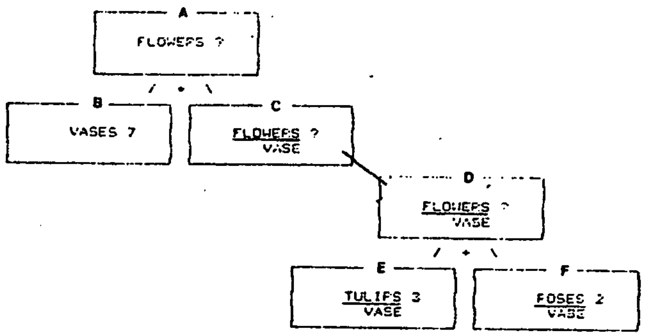
Problem 3: There are 7 vases, with 3 tulips and 2 roses in each vase. How many flowers are there in all the vases?



Note the location of the question mark. The calculation for the solution for Problem 3 will be (E+F)xB. The latent component in both problems is the same and is shared by both underlying structures. The two problems also describe the same situation in the world. However, Problem 2 calls for division and subtraction while Problem 3 calls for addition and multiplication. Of course the route to the solution is different for Problems 2 and 3. In Problem 2 we began the solution with the multiplicative structure, but in Problem 3 we start with the additive structure. Working with schemes makes it clear where to start (what is the first operation). One always starts from the structure that has two complete components, and whose output will become part of the structure that initially had only one complete component.

Types of Schemes for Two-Step Problems

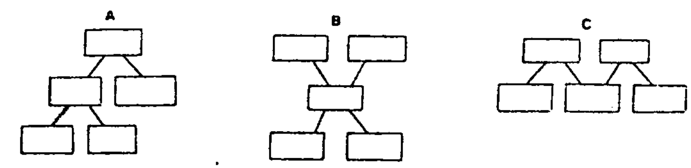
Given that any scheme (for two-step problems) consists of two structures, there are in principle only three ways in which the two structures can be connected. If any additive structure consists of three components, the whole, and its two parts [and a multiplicative structure consists of the product (parallel to he whole) and its two factors (parallel to the parts)], then the three combinations produce the following Schemes:

Scheme A The whole of one structure becomes a part of the other structure (Examples 2 and 3 above).

Scheme B The two structures share one whole.

Scheme C The two structures share one part.

Figure 3 presents graphically the three basic combinations possible of two structure. We use Shalin's graphical notation here, since it is a well-known representation also used by Resseur (1990).



Empirical Findings

We conducted an empirical study to determine the validity of categorization by schemes, in comparison to categorization by operations. In the study we tested 21 types of problems that combine one additive structure with one multiplicative structure. The 21 problem types are all the possible problems, where a distinction is also drawn between intensive and extensive quantities in multiplication. The 21 problems were presented in four contexts each. The test was divided into eight different questionnaires and administered to 253 children in grades 3,4,5 and 6.

The variables were: context (with 4 values); schemes (with 3 values; operations (with 12 values - ax(b+c); ax(b-c); a:(b+c); a:(b-c); (a+b):c; (a-b):c; a+(bxc); a+(b:c); a-(bxc); a-(b:c); (axb)-c; (a:b)-c); and whether the quantities were intensive or extensive (2 values). The dependent variable was the success percentage for each problem as a measure of its difficulty level.

Findings: Table 1 presents the mean for each scheme and each order of operations for all contexts (since there were no significant context-related differences).

Table 1: Success percentage by scheme and order of operations in all contexts

1st operation	2nd operation	Scheme A	Scheme B	Scheme C
+	×	83	–	–
×	+	73	–	–
+	÷	72	72	–
÷	+	–	–	60
×	–	72	55	–
–	×	–	–	60
–	÷	66	–	37
÷	–	55	–	38

Analysis of variance produced significance for the variables: schemes ($F = 162.24$, $p<0.000$); order of operations ($F = 213.07$, $p<0.000$) and we are now investigating the interaction between schemes and operations. The variable of intensivity vs. extensivity was also significant, problems with intensive quantities being harder than problems with extensive quantities (56% vs. 66%, respectively). The context did not yield a significant effect. (Additional data will be presented at PME.)

References:

Gray G.B. & Young R.V. (1940): "The effect of process sequence on the interpretation of two-step problems in arithmetic", Journal of education research vol 34, No. 1

"One, two and... three" (1975) - Textbook for elemantary school for grade three.

Schwartz J.L. (1986): The Algebric Proposer a mathematical enviroment for analysis, modeling & problem solving.

Shallin V.L. & Bee N.V. (1985): "Structural Differences Between Two-step Word Proplems". Presented at the Meeting of the American Educational Research Association.

EARLY CONCEPTIONS OF DIVISION. A PHENOMENOGRAPHIC APPROACH

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The question concerning primary schoolers' capabilities to understand division as the reverse of multiplication, without exploring the difference between quotitive and partitive division, has long been debated, at least in Sweden. However, examples from a study addressing pupil's conceptions of division before it is taught, illustrate that children already are aware of the relation between multiplication and division, as well as of the relations and differences between quotitive and partitive division. The children could solve both these forms of division through proportional thinking. The strategies through which this conception of division was observed were of a multiplicative kind. They gradually seemed to develop into early conceptions of the commutative, distributive and associative qualities of multiplication, into knowledge of the multiplication table, and finally into a conception of division as the reverse of multiplication.

Aims

An earlier study addressed conceptions of number and addition-subtraction strategies developed informally by 7-year-old children in Sweden before they began school (Neuman, 1987). A small part of this study was presented at PME XIII (Neuman, 1989).

The presentation made here is part of an ongoing study in which an attempt has been made to map out children's conceptions of division and fractions. 'By contrast with the work on addition and subtraction, most of the research on multiplication and division has been done with children in the 12 – 14 age-range' Greer states (Greer, 1987, p 65). Further, the research on multiplication and division to a large degree has concerned the choice of operation and not the strategies chosen in order to calculate the solution, he adds. The work on division presented here is done with younger pupils in the 7 – 13 age range, most of them 7 – 10 years old, and it primarily concerns calculation strategies and conceptions of division.

The phenomenographic approach

In both the earlier and the present study a phenomenographic research approach has been adopted (Marton, 1981), which means that neither factual knowledge and procedural skills per se, nor the developmental levels of children in Piagetian terms, have been of interest. In phenomenographic investigations the results are descriptions of *the conceptions themselves* and the relations between them. 'Conceptions' in a phenomenographic sense mean the relations created between individual and world, that is, the understanding developed through individuals' experiences of the world. In particular, this presentation explores the relations between the conceptions of division, and the procedures through which they are made observable, and with which, in informal learning situations, they are developed in close mutual interdependence.

It is assumed, and illustrated through many studies especially of children, that the conceptions of the subjects who are interviewed might be so different from those of the researcher that it is not possible to construct hypothetical models of the conceptions that will be mapped out before the study is carried out (see e g Neuman, 1989). Yet, afterwards it is often possible to compare the results to conceptions described in history, to those found among people in non-Western countries, and, of course, to results found by other researchers.

Design of the study

Silver (1986) illustrates that difficulties in division might be caused by the fact that pupils experience division mainly as partitive, while the division algorithm 'is typically understood, if at all, from a quotitive point of view' (p 196). Further, teachers and teacher educators, at least in Sweden, have been concerned that children's informally developed division strategies are not taken into account when division is introduced as the reverse of multiplication, without focusing on differences and relations between quotitive and partitive division. This practice is now common in Sweden. The main aim of the investigation referred to here was to find out about what these early strategies, and the conceptions they expressed, might look like.

Of the two problems to be dealt with here the first was a partitive and the second a quotitive division problem. They were presented as follows:

1. Four boys have got 28 marbles to share. How many marbles does each boy have?
2. Mum has baked 42 buns. She puts them into plastic bags, six in each bag. How many bags does she need?

The investigation was carried out in a main study and a follow-up study in three different communities in Sweden. The follow-up study was performed (for reasons that will not be explored here) in order to see what happened when the numbers in the problems were altered so that there were 7 boys sharing the 28 marbles, and 32 buns of which 4 were put into each bag.

The two questions were given to pupils in grades 2, 3, 4 and 6, alltogether 71 pupils.

In this presentation I will give neither an account of the many answers from pupils who solved the problems through a known multiplication fact nor of the few answers given through a rough estimation, e g 'Eleven I think ... or nine ...'. What mainly will be described here are conceptions I have called 'original', since they were informally developed.

Thirty pupils expressed original conceptions when they solved problem 1 (19 from grade 2, 8 from grade 3, 3 from grade 4) and forty when they solved problem 2 (19 from grade 2, 12 from grade 3, 8 from grade 4 and 1 from grade 6). Only one original conception, one which was related to proportional thinking, was expressed when the quotitive division problem (2) was solved. This conception was the most frequently expressed even when the partitive division problem (1) was solved. However, beside it another conception expressing a 'dividing up' way of thinking, was identified for the partitive division problem.

The 'Proportional' conception related to the quotitive division problem (2)

The pupils solved the quotitive problem by trying to find out the proportionality between number of buns and number of plastic bags: 6 (4) buns – 1 bag; 12 (8) buns – 2 bags, and so on. The conception they expressed has therefore been labelled 'Proportional'. The strategies through which the 'Proportional' conception was expressed, and which were demonstrated in drawings as well as in mental calculation, were:

1. Repeated addition or subtraction
2. Repeated doubling
3. Transformations
4. Distributive multiplication

The drawings mostly were made by the second graders. They drew one bag at a time, putting 6 (4) buns into it (addition), or 42 (32) buns, encircling six (four) at a time (subtraction). Two pupils drew more than one bag and put 6 (4) buns into each of them to begin with. After having counted the buns they drew one bag at a time putting in 6 buns until they reached 42 (32; 'distributive multiplication').

The mental strategies mirrored the drawings. However, of the 9 pupils who tried repeated addition mentally, only 3 managed to arrive at a correct answer, and none of the 3 trying repeated subtraction did so. It is difficult to keep track of as many sixes or fours as the ones occuring in the problems, and the supporting notes and finger counting strategies the pupils had to use were complicated.

However, of the 15 pupils who tended to structure their thoughts 12 managed to solve the problems correctly without any advice. One pupil mentally transformed a 10-grouping into a 4-grouping, explaining: 'I take 2 fours out of each ten ... 6 fours (in 32) ... then there are 3 twos left within the tens ... and so the 2 ... (the two units) ... 8 bags'. The other 14 pupils used the distributive qualities of multiplication. In the follow-up study, where there were 32 buns 4 in

each bag, one pupil knew that 6 x 4 = 24 and then added 4 + 4, answering '8 bags', and another began with his knowledge of 10 x 4 and subtracted first 4 and after that 4 more. In the main study, where there were 42 buns, 6 in each bag, four pupils knew the 'multiplication double' 6 x 6 = 36 and began with that, adding 6 to 36 before they answered '7 bags'. Five began with 4 x 6 = 24, or with 12 + 12 = 24, and added 6 three times in order to have 42 buns. One knew that 12 + 12 = 24 and that 24 + 24 = 48 and subtracted 6 and one doubled 6 x 4 = 24 (note: he did not say 4 x 6) and then subtracted six. Two more began with their knowledge that 6 x 8 = 48 (note: again not 8 x 6) and subtracted six. One pupil even drew his conclusion that there had to be 7 bags, each with 6 buns, from his knowledge that 6 x 7 = 42, since 7 x 7 =49! Had these pupils begun to understand something of the commutative quality of multiplication?

The strategies through which the 'Proportional' conception was expressed were of a multiplicative nature, except the 'subtractive drawings' and the 3 repeated subtractions strategies with wrong answers. The difficulty of keeping track of the numbers led the children to create less cognitively demanding multiplicative strategies which finally seemed to end up in knowledge of the multiplication table (see compatible results presented by ter Heege, 1985). This knowledge was closely related to early conceptions of the commutative, distributive and also of the associative qualities of multiplication. To 'double', for example to multiply 2 x 24, in order to find out that 8 x 6 = 48 can be seen as making the known factor, 6, four times larger and after that again to 'redo' the multiplication as 8 (4 x 2) times 6 (1/4 of 24). Yet, the children, of course, just 'doubled'.

The 'Proportional' conception seemed to have the potential to develop into a conception of division as the reverse of multiplication, and at the same time to prepare the way for conceptual and factual knowledge of multiplication.

The two conceptions expressed in partitive division (problem 1)

It is easy to understand that pupils are able to carry out quotitive division through repeated drawings or encircling of 6 (4), and later through repeated addition or subtraction and so on. It is harder to understand how partitive division can be carried out before children know the multiplication table. Repeated addition or subtraction is hard to carry out if you do not know what to add or subtract, and that is just what you do not know in partitive division, e g when 28 marbles are to be shared out equally between 4 (7) boys.

However, young children who have not been formally taught division seem to believe that it is possible to solve all problems in some way. As mentioned earlier, two original conceptions of how the partitive division problem could be solved were mapped out.

The 'Dividing up' conception expressed in partitive division

The first conception of partitive division has been called 'Dividing up'. It was observed through three different strategies demonstrated in drawings or in mental calculation:

1. Repeated estimation 2. Repeated halving 3. Distributive division

Pupils using 'Repeated estimation' encircled the interval within which the searched number could occur, in order finally to reach the correct answer. Four pupils drew for example first 9, then 5 and finally 7 marbles four times, before answering '7 marbles each' (the follow-up study). One pupil added those numbers four times and one multiplied them by 4.

The two pupils who carried out repeated halving first divided 28 into 'two parts' and after that divided 14 into two parts. One of them made a drawing first.

The four pupils who used the distributive qualities of division all begun to divide up 20, stating that the four boys had got five marbles each and then from the 8 remaining marbles each boy got two more and that they thus got seven marbles each.

All those strategies were division strategies, even if multiplicative strategies were used in order to control the estimations made in 'repeated estimation'. 'Repeated estimation' was the only one of the three strategies related to the 'Dividing up' conception which had general applicability. However, unlike the multiplicative strategies related to the 'Proportional' conception, the multiplicative strategies used to control the repeated estimations were laborious since they had to be carried out many times.

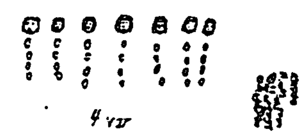
The 'repeated halving' strategy has limited applicability. It can only be used if the dividend is 4 or 8. This is also the same for the 'distributive division' strategy, which only occasionly can be used before the multiplication table is known. For example, this strategy is hard to use without knowledge of the multiplication table if 56 marbles were to be divided up between 7 boys.

The 'Proportional' conception expressed in partitive division

Only 12 pupils expressed the conception 'Dividing up' when they solved problem 1, the partitive division problem, while the remaining 18 pupils expressed the 'Proportional'

conception and used the same strategies as those used in problem 2 even when they solved problem 1.

The drawings for repeated addition also could be of the same kind: 28 marbles were drawn and after that 4 (7) at a time were encircled. However, two 'repeated addition' drawings were of the kind that is illustrated here by Jenny:



Jenny makes a matrix. First she draws the seven boys and the 28 marbles. After that she 'takes' one marble at a time, through marking it, and shares it out to one boy at a time. When the first seven marbles are shared out she begins with the first boy again, sharing out seven more marbles and after that the procedure is repeated twice more. In the end Jenny directly states that each boy gets four marbles.

It is possible to observe the proportional qualities built into Jenny's drawing. Each time seven marbles are distributed, each boy gets one marble. Since the distribution is repeated four times, they get four marbles each.

Two pupils in the main study tried to carry out repeated addition mentally. None of them managed to keep track of the many fours they then had to add. Still, Emma (grade 2) clearly illustrates the 'Proportional conception', saying when she has uttered the word 'four' that the boys now have got 'one each', when she has uttered the word 'eight' that they have got 'two each' and so on. One pupil in the follow-up study, where there were 7 boys sharing the 28 marbles, easily solved the problem through 'Repeated doubling' explaining that he first 'took 7+7' and after that '14 + 14' and that each boy then had 4 marbles. When asked about what he did with the seven marbles when he had 'taken' them, he answered that he 'put them into a hip', before he shared them out one to each boy. The pupil who solved problem 2 through a 'transformation' solved problem 1 in the same way, transforming a '10-grouping' into a '7-grouping', explaining that she 'took one seven out of each of the 2 tens ... then there were 2 threes ... six ... left (within the tens) ... and so 8 (units) ... 2 sevens more ... that's 4 sevens'. She also explained, when asked what happened with the 'sevens' which she 'took' four times, that she shared them out one to each boy when she had 'taken' them.

Eight pupils used the distributive qualities of multiplication, 3 of them through 'matrix-drawings'. The difference between these 'matrix-drawings' and for example Jenny's drawing was that these 3 pupils began to share out 4 or 6 to each boy and after that shared out one at a time. The pupils using mental strategies of this kind could think: '4 x 6 .. 24 .. 4 more .. 28 .. 7 marbles each.' Thus, these pupils began with an estimate but chose in the end a strategy related to the 'Proportional' conception, instead of making a 'repeated estimation'. Their conception of division has therefore been categorised as 'Proportional'.

Discussion

The fact that more pupils solved the partitive division problem using strategies related to the 'Proportional' conception than using strategies related to the 'Dividing up' conception, points to the fact that in their informal thinking, and in the problem solving they perform through drawings, children can conceive of the two forms of division in the same way, and use the same strategies for solving them, while still being aware of what question they should answer.

The sharing out of one at a time, which was found in 'encircling', in 'matrix-drawings' and also in mental calculation strategies, expresses the most original conception of division as carried out in real situations (see e g Miller, 1984). Thus, one might expect that all of the children, even those expressing the "Dividing up" conception for problem 1, might already have had the 'Proportional conception' at hand for both kinds of division. If not they should easily be able to realise that strategies related to the 'Proportional' conception could be used for all kinds of division, if they were given division problems to solve when division was introduced, and were given the opportunity to discuss the drawings or the mental strategies they had used when they solved these problems. Strategies related to the 'Proportional' conception, with its great potential for development of the conception of division as reversed multiplication, would then be the one that made all kinds of division problems possible to solve. Further one would expect that the multiplicative strategies related to the 'Proportional conception' would help all pupils to develop knowledge of the multiplication table closely related to conceptions of the commutative, distributive and associative qualities of multiplication.

Instead of helping children in elaborating their own informally developed mathematical thoughts, algorithms are introduced early in traditional teaching. The result of this teaching

twas illustrated in the investigation referred to here. Three of the answers from the third graders and 5 of those from the fourth graders were the result of using the subtraction algorithm: the pupils had put 4 under 24 or 6 under 42, answering 20 or 36 respectively. Many of them did not even question the answer. Further, of the 22 answers given by the 11 pupils in grade 6, 4 were solved through the short and 3 through the long division algorithm. Only 3 of these answers were correct. The 8 pupils using the subtraction algorithm, and the 3 using the division algorithm without obtaining a correct answer, did not seem to have developed any mental calculation strategies or (probably because of that) to know the multiplication table. Nor did they seem to understand the operations they carried out.

A question that spontaneously arises from these observations concerns how early – if at all – pupils have to learn algorithmic procedures they will never use as mini calculators and computers will do the calculations they will need in their adult lives. The development of heuristic strategies which will be of great importance as a supplement to computerised calculation seems to be seriously obstructed by teaching algorithms before those strategies are developed to the point where they have become powerful tools of thought. This kind of teaching seems to do greater damage to the development of conceptions that are closely related to procedures and factual knowledge than would the introduction of division as the reverse of multiplication, especially if this introduction was done in such a way that the pupils together created this knowledge, setting out from the problem solving strategies and conceptions they already have developed informally.

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References:

Greer, Brian (1987). Understanding of arithmetical operations as models of situations. In J Sloboda & D Rogers (Eds) *Cognitive processes in mathematics*. Oxford: University Press.

ter Heege, H (1985). *The Acquisition of basic multiplication skills*. National Institute for Curriculum Development, Enschede, Netherlands, Paper presented at PME X.

Marton, Ference (1981). Phenomenography – Describing the world around us. *Instructional Science, 10, 177–200*

Miller, K. (1984). Child as the measurer of all things: measurement procedures and the development of quantitative concepts. In C. Sophian (Ed), *Origins of cognitive skills*. London: Lawrence Erlbaum.

Neuman, Dagmar (1987). *The origin of arithmetic skills:. A Phenomenographic approach.* Göteborg: Acta Universitatis Gothoburgensis.

Neuman, Dagmar (1989). Early conceptions of subtraction, *Proceedings of the PME-XIII conference*,. Paris.

Silver, E. A. (1987). Using conceptual and procedural knowledge: A focus on relationships. In Hiebert, J (Ed) *Conceptual and procedural knowledge: The case of mathematics*. Hillsdale, NJ: Lawrence Erlbaum

CAN EPISTEMOLOGICAL PLURALISM MAKE MATHEMATICS EDUCATION MORE INCLUSIVE?

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There is growing concern about the exclusive nature of higher achievement in mathematics in schools at a time when there is an increasing use of computers in schools. In this paper, a course for undergraduate primary teachers is explained and the theory that the friendliness of the computational environment can make it a useful remedial mathematics medium is examined. What has been termed 'epistemological pluralism' by Turkle and Papert (1990) is advocated as an alternative approach to the more traditional one which aims to help students reach an intellectual maturity which is based on the graduation from concrete to abstract processes for the construction of knowledge.

There is a continuing problem with what might be called inclusive mathematics education in schools. Many students are alienated from mathematics by barriers which lie outside the domain of mathematics itself, things such as cultural cringe (as it is known in Australia) can make students who profess positive attitudes to mathematics part of a cultural 'elite' which soon becomes a rejected minority in schools.

While not suggesting that an isolated incident or comment can make a difference, we are all conscious of times when students have been brought to reassess their 'mathematical shyness' and with new-found confidence work to achieve new heights of personal success. The problem confronted in the course reported in this paper was how to provide experiences for a large number of students which would have this effect.

The theory underlying the course reported was that if a group of 'math shy' undergraduate primary teachers could be given positive reinforcement of their own abilities and a more mathetic perspective (see Higginson, 1972) from which to approach mathematical activity by working at mathematics in a suitably constructed computer environment, they would learn more mathematics as undergraduates and be better equipped to help fight the mathematical poverty of the schools in which they would be working. The course did not rely simply on providing 'nice' experiences. Logo programming was used to provide a formal, concrete environment in

which to do mathematics in order to foster or rekindle interest in mathematics and understanding of mathematical processes in a curriculum which depended on a surrounding culture of 'epistemological pluralism' (Turkle & Papert, 1990). This did not mean that students could avoid the formality of mathematics but it did mean that they could work legitimately as bricoleurs, piecing together and polishing their work as apprentice craftspeople.

The aim was to provide the students with an alternative path to achievement of mathematical products, not to lower the standard of the mathematics in any way. Papert and Turkle argue in a paper about similar problems within the domain of computer science (Turkle & Papert, 1990) that this demands the rejection of the notion of bricolage as an immature thinking process and the acceptance of it as an alternative process. They argue that computer virtuosos frequently employ concrete processes to achieve their high standards and that the preference for the use of one style of knowledge construction in preference to the other should no longer be considered inferior.

Analyzing the data collected from 60 students during a course which promoted the legitimacy of mathematical bricolage provided an opportunity to evaluate the conjecture reported recently by Turkle and Papert (1990 p. 153):

> The development of a new computer culture would require more than environments where there is permission to work with highly personal approaches. It would require a new social construction of the computer, with a new set of intellectual and emotional values more like those applied to harpsicords than hammers.

They, and we, have chosen to adopt the definition of bricolage used by Levi-Strauss (Turkle & Papert, p 136),that:

> Bricoleurs construct theories by arranging and rearranging, by negotiating and renegotiating with a set of well-known materials.

Initially, the 60 students in our course responded to a survey of attitudes to mathematics and computers. Then there were weekly discussions about the nature of mathematics and mathematical problems were discussed for their diversity of subject-matter (from areas such as information theory, chaos, number systems and cultural differences); the students spent one and a half hours working in Logo microworlds per week, often returning to the same problem-domains for three or four consecutive weeks; the students wrote weekly journals in which they reflected upon their experiences in the course and their previous experiences; some students were video-taped during working sessions and all were audio-taped during a final discussion of the nature of mathematics, and each student submitted a final essay relating their current understanding of the nature of mathematics and its relevance to them with their previous understanding, as reported in their journals throughout the semester. In addition, the students weekly submitted activity sheets on which they proposed activities for young maths students.

The following is a summary of the information gathered about the students' attitudes before the course started. There were 35 questions in the survey and of the total sample of 60, 13 students were boys.

ATTITUDE TO COMPUTERS:

Computers would be interesting at home; I'd like to learn to use one, it would be easy, I'd use it more than other members of my family, I enjoy computing and mathematics would be more fun with computers

and it is not relevant that they were not necessary in the past, not true that they are not good for the world, not fun, might get too powerful, or would be hard for me to use.

More than 60% agreed, 30% were not sure and 8% disagreed.

ATTITUDE TO MATHEMATICS

I do extra work, it's easy, it's important that I understand, and I feel good about it

and it is not the case that I feel tense, don't do well, have to remember things, get upset, can't understand and only do mathematics because I have to.

Almost 40% agreed, 20% were uncommitted and nearly 30% disagreed.

SOCIAL ATTITUDE TO COMPUTERS

Smart, science-type people like computers and they are not very sociable or athletic.

Only 3 students agreed with these statements but 25% were not prepared to say they were not true.

In the course at Harvard as reported by Turkle and Papert, the students identified computers with mathematics and those who were mathematics shy were similarly computer shy. Many of these students, according to the conjecture, while not denied physical access to computing, were loath to traverse the mental barrier they perceived to lie between them and computing, the knowledge construction techniques of the canonical elite.

In a large survey of Canadian students (Collis, 1988), it was found that the majority of students associated computing with a masculine, mathematical model and for many lower-achieving girls, in particular, studying computing was a daunting prospect. The aspects of computing which were said to contribute to this negative attitude were school policies and practices; social expectations, and personal factors. Collis et al stated that in Canada, where at the time students did not use computers for instructional purposes or within a broad range of curriculum areas as tools, the students do not develop an image of it as something which will be of interest/value to them and (Collis, 1988, p. 122):

> Through this omission, adolescent females as well as males are deprived of access to many valuable aspects of constructive computer usage.

The Canadians concluded that of the major changes necessary, more scope for collaborative work was one and gender-typing another. In order to correct the imbalance, the report suggested that it would be necessary to change teacher education and school practice to widen the range of experiences and use the computer as a tool in the curriculum; to organize the computer use better to make it more equitable, and to counsel the girls who

don't like computers. Many of these changes have been tried in schools and it is their continuing failure to achieve all the results desired that brings our attention to the ones advocated in this paper. In our work with computers we have come to believe that the problem goes deeper and that the focus must be on what sort of tool the particular user is likely to be able to use best, on whether beyond what Turkle has called 'hard mastery' (Turkle, 1984), which for some is perfect, there lie other types of usage which are equally successful in the hands of hitherto non-participating users.

In continuing work relating to the introduction of computers into education, we have developed a series of metaphors for the representation of computers as tools and we have found they operate with varying success. We prefer to think about how students are to be encouraged to use the computer than in within which particular disciplinary area.

In the Melbourne course, the situation was very different from that reported by Collis et al and Turkle and Papert. By 1988, Victorian schools had been persuaded by and large to use computers across the curriculum, particularly in subjects such as history, geography and language, as well as for vocational training for those students who were to remain at school only until they could gain employment. The survey shows that the students in the course did not agree with the Harvard students about computers. The few who had managed to stay within the mathematics streams at their schools had rarely used computers and knew that if they did it would be to work on programs and problems which they tended to associate with mathematics and science. They did not think programming was a trivial exercise. The majority of our students, more likely to be lower-achieving mathematically and so more involved in word-processing classes, had a different attitude. Any programming they had done would most probably have been elementary Logo programming and most of their experiences would have been designed to convince them that mastery of the computer is easy. It is not surprising that these students considered programming to be easy and likely to cause them no particular problems despite their difficulties with mathematics. This meant that for our students, using the computer and Logo was not likely to pose a problem which they would associate with previous failure in the same way as more traditional mathematical activities in a traditional medium would have done. That was our contention and the premise upon which the course was designed.

In our course, the students worked in Logo in microworlds which focussed on mathematical facts related to numbers and geometry. The activities they undertook were aimed to have them participating in, and so concretely representing, mathematical processes such as hypothesizing and proving their conjectures, often doing this by particularizing and generalizing, and collaborating with other students to negotiate problem posing and solving methods. Mathematics was promoted throughout the course as dynamic knowledge: it was said to entail both the processes and products of mathematical activity. The students' set-work involved the investigation of problems which they developed from worksheets such as the one shown:

how many sides do stars have?

what theorems are there?

what is a lemma?

```
TO STAR :ANGLE :SIDES
REPEAT :SIDES [FD 50 RT :ANGLE]
END
```

is there a general rule?

are stars polygons?

are there some special circumstances?

what connects :angle and :sides?

what :angle does it need?

The unusual aspects of the activities for these students included the following: the sustained nature of the work; the open-endedness of the work; the freedom of choice with respect to the processes of working and the accompanying lack of direction, which led to a diversity being developed within the group to which common standards essential for such processes as proof could be applied meaningfully; the collaborative nature of mathematics within this culture; the concrete nature of the mathematical processes; the friendliness of the Logo environment which is so instantly forgiving of corrected mistakes compared to the white-page medium where errors are for ever; and the graphical nanire of the work which contributed to the students' appropriation of problems as given and often resulted in personally meaningful decoration being added to work in progress.

In the words of the students:

> In the Sunrise School we were not given a bunch of mathematic facts or sums as I had been so used to in primary and secondary school which had led me to believing that's all mathematics was. That's the reason that I was so surprised to learn of the huge amount I took part in each day. We were encouraged to find things out on own, to take risks and to experiment with the little information that we were given. I found this to be an effective way of teaching mathematics as we were not given a question-answer situation where there was only one right answer. Each person discovered different methods and worked at their own pace. No one knew exactly what the teacher expected them to come up with at the end of each lesson so we were made to do the thinking ourselves and to work out our own problems that we created out of interest.

The result was that new ways of learning were soon on the agenda:

> At school I always learnt geometry by the process of problem solving and an endless amount of writing. However, I still never fully understood it. Learning how is difficult, but learning why ... seemed impossible. The Sunrise school seemed to change my ideas of geometry by converting problem solving activities to an activity where I was able to actually see the formation of angles according to different shapes. Turtle geometry allowed me to change numbers in a fixed program and allowed me to actually see up on a screen the value that they numbers had. By using a simple program such as to polygon sides, repeat sides forward twenty left 360 divided by sides end and placing numbers where appropriate I was able to work out different shapes of different sizes and therefore different angles. As I was able to see the shapes forming and relating the shapes to the numbers I used, geometry became understandable.

The students were not all immediately post-schooling and one of the mature students showed all the typical signs of anxiety of a mother returning to the work-force: teacher training was a long way from dress-making. She wrote in her journal after a couple of weeks:

> Logo presents as an easy step by step method of programming. When I was first introduced to Logo, I could not have been more intimidated by the prospect of having to learn this strange new skill. However, after a few weeks of practice it became clear that this computer language has many parallels with simple logical task solving. For instance, in using Logo turtle graphics I began by running a polygon generating formula of which I had no direct understanding. This lack of understanding left me quite frustrated at first and I was afraid of not ever being able to understand. It was not until a closer examination of the formal steps described in the formula itself that I gained the necessary insight to create a specific formula of my own. Logo, like common household and work related tasks, was seen to involve fundamental concepts of practical, logical, and step by step thinking. ...
> This course has revealed not only the broad, wide-ranging applications of mathematics generally, but also the relationship between virtually every task and the mathematical thinking I once thought was the sole province of mathematicians, accountants, engineers and the like. As a consequence of this revelation I have realised the importance of gaining a deeper understanding of the mental steps involved in virtually every practical task. This experience has enhanced by appreciation of the subtleties of the teaching process.

For the students it was important to provide a culture which would scaffold them as they came to terms with their own understanding:

In her second week, one student wrote:

> Great satisfaction!! These are two words to describe my feelings. Although I've used computers for school work before, I never realised the full potential of their functions!! They are created with the use of a few signals to the computer a series of shapes of different sizes and colours on the computer display. It's wonderful to experience a certain amount of control over what is displayed on the screen. My satisfaction, is only a small portion of the total feeling in the room, however. Everyone had created images hence today I learnt that knowledge is the key to control.

Later she added:

> Suddenly my perfect uncomplicated image of maths is changed from black and white to an array of psychedelic splashes of colour. Suggestions of dots being potentially rich ... stunned me, that is until I looked at my eight year old cousin's colouring book, aha, I cried, dot to dot activities. Yes I realise their potential, number recognition, number sequences had started to make amazing sense to me and I had begun to critically investigate my current concepts of maths but also my childhood experiences in maths.

Many of the students reported their interactions with other students and throughout the semester they became critically aware of the value of good collaboration:

> I yearn to be able to understand everything I can about mathematics. I desperately want to be able to teach the children well and with confidence. So far I have become more aware of the way my mind works and using computers in the course the way we have been has helped me to learn how to just sit and figure problems out. I now have more of a desire to want to sit down and figure out a problem just for my own satisfaction. My mind has become more open and more open to thinking up ways of solving these problems. I am now able to put more pressure on myself to find out something because I am more confident that I can finally reach an answer. The activities on the computer has helped proved this to me. As time went on I was able to offer more suggestions as to why things were so and eventually both of us (my partner Anna) benefited from each other's gaining and growing knowledge.

But the aim of the course was to give the students opportunities to work on their own personal meaning of mathematics. The introspectivity fostered by the course and recorded by the students was most revealing:

> This course I've attempted to talk about maths and the problems I have had. At the beginning most of my discussions were attempting to get a better understanding about the course. I don't think that I have ever

talked about maths as much as I did at the start of the semester. Towards the end I began to talk about maths and how it effects me, the type of things I do and how maths relates to my everyday living. I have also noticed the difference in my attitude towards numbers. I didn't place great emphasis on the importance of numbers and how much children need to understand them, but doing and learning different activities made me see that numbers can be enjoyable. Children should be able to experiment and this could lead to talking about maths using everyday language.

The aim of the course was to use computers to bring students back into the mathematical culture to enable them to work productively in the future in a field from which they had been alienated, in many cases officially for lack of achievement and in others, frequently by barriers built of negative associations and alien practices. The evidence showed that there were many students in the group who preferred to work closely with objects in what increasingly is being described as a typically feminine way. We prefer to associate this style with craftspeople and virtuoso artists. We suggest that since the introduction of computers into schools, providing a concrete medium in which bricoleurs can work at mathematics, there is no further excuse for losing from mathematics those students who prefer to work in this way.

References:

Collis, B.J., 1988. "Adolescent Females and Computers: Real and Perceived Barriers" in (eds.) J.S. Gaskell and A.T. McLaren, *Women and Education A Canadian Perspective*, pp 117 - 131. Alberta: Detselig Enterprises Ltd.

Higginson, W., 1972. Towards Mathesis, unpublished PhD thesis. Alberta: University of Alberta.

Turkle, S., 1984. *The Second Self: Computers and the Human Spirit*. New York: Simon & Schuster.

Turkle, S. and Papert, S., 1990. Epistemological Pluralism: Styles and Voices within the Computer Culture in *Signs: Journal of Women in Culture and Society 1990*, vol.16, no.1. Chicago: University of Chicago.

THE PUPIL AS TEACHER:
ANALYSIS OF PEER DISCUSSIONS, IN MATHEMATICS CLASSES, BETWEEN 12 YEAR-OLD PUPILS AND THE EFFECTS ON THEIR LEARNING

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After a brief description of the main project out of which this paper arises, we discuss the methodology of theoretical sampling and the process of analytic induction by which we derived the categorisation of our data. We then focus on one particular category of data which we call 'Pupil as Teacher' and look at some of the inferences which can be abstracted from the protocols within this category.

This paper arises out of a project entitled "Mathematical discussion - is it an aid to understanding?" This all embracing title obviously required some specialisation within the area and what we have been looking at is pupil - pupil discussion within the classrooms of teachers who consciously and deliberately use discussion as one of their teaching styles. A detailed description of how the project was initially set up has been given in Pirie and Schwarzenberger (1988a). Here we intend only to give a brief outline of the features which are relevant to the particular data which we intend to examine in this paper. The data were collected by tape-recording small groups of pupils as they worked together in their lessons and these were supplemented with field notes made by an observer in the classroom, who recorded activities on a time-sheet which could subsequently be matched to these recordings.

Before we could begin any analysis of the data, we needed to create an explicit definition of 'discussion' by which we could identify those episodes that we should be examining. In Pirie and Schwarzenberger (1988a) we put forward the following which has remained our working definition ever since then. Discussion is:

- Purposeful talk ie. there are well defined goals even if not every participant is aware of them. These goals may have been set up by the group or by the teacher but they are, implicitly or explicitly, accepted by the group as a whole

-On a mathematical subject ie. either the goals themselves, or subsidiary goals which emerge during the course of the talking are expressed in terms of mathematical content or process

-In which there are genuine pupil contributions ie. input from at least some of the pupils which assists the talking or thinking to move forwards. We are attempting here to distinguish between the introduction of new elements to the discussion and mere passive responses such as factual answers to teachers' questions

- And interaction ie. indication that the movement within the talk has been picked up by other participants. This may be evidenced by changes of attitude within the group, by linguistic clues of mental acknowledgement, or by physical reactions which show that critical listening has taken place, but NOT by mere instrumental reaction to being told what to do by the teacher or another pupil.

It became abundantly clear from the first set of data that we collected, that there would be no simple answer to the research question we had posed - nor indeed had we expected one. We had examples both where discussion appeared to advance the understanding of the pupils involved, and where it quite definitely confused and misinformed the participants and inhibited progress towards the solution of their problems. Our task, therefore, became one of looking for meaningful ways in which we could talk about the data in finer detail and thus produce indicators of fruitful or unfruitful discussion.

There has been much research into the effects of discussion on learning, (Barnes and Todd 1981, Webb 1982), in particular teacher - pupil discussion, (Mehan 1985, Barnes et al 1986), but none that we have been able to identify on the effects of peer discussion on mathematical understanding. Since one of our hypotheses is that the situation in mathematics learning is arguably different from that in other disciplines, we did not wish to pre-judge the outcome of the analysis by imposing what

might be inappropriate categorisations of the data. We thus turned to the process of theoretical sampling to inform our further data gathering. The data collection was to be "controlled (sic) by the emerging theory" (Glaser and Strauss 1968, p 45). Our intention was to use analytic induction to abstract relevant, essential characteristics of the discussions from concrete cases, rather than use statistical sampling and enumerative induction to produce categories based on their generality. "Enumerative induction abstracts by generalisation, whereas analytic induction generalises by abstracting." (Znaniecki 1968)

On first analysis three features seemed to stand out as being of possible interest. The first of these was what it was that gave the speakers something to talk about. Within this feature the episodes could be classified into groups characterised by whether

(a) they had a task or concrete object as the focus of their talk;

(b) they did not have an understanding of something, but knew this and thus had something to talk about;

(c) they did have some understanding and that gave them something to talk about.

The second feature was the kind of language used; the focus being on the language in which the discussion was conducted and not on the content of the statements made. Again there were three categories which suggested themselves. Those where

(f) the speakers lacked appropriate language; they did not have the correct or useful words;

(g) the speakers used ordinary language;

(h) the speakers used mathematical language.

It could be suggested that the categorisation of language as 'ordinary' or 'mathematical' would be somewhat arbitrary since 'mathematical' language for young children might be 'ordinary' language for them a few years later In practice, however, viewing the discussion in the context in which the

pupils were working enabled us to make decisions with little difficulty, although some subjectivity was inevitable.

The third feature on which we focused was the kind of statements the pupils were making. There could be a variety of statements within any one episode. These we classified as

(p) incoherent - that is to say incoherent to us, the observers;

(q) operational, or, in other words, about specific (frequently numerical) examples of mathematics;

(r) reflective - we have subsequently re-named this category 'abstractive' as its nature became evident more in terms of statements of generalisations of mathematics than in terms of statements reflecting upon mathematics.

Examples of the use of this categorisation can be seen in Pirie and Schwarzenberger (1988a) and Pirie and Schwarzenberger (1988b).

Having thus evolved a tentative working categorisation of the data, we proceeded to observe further small groups of pupils. We tried to analyse this second batch of data from a different perspective, namely from that of the behaviours of the pupils, looking at both their mathematical behaviour and the roles they verbally took up within the group. At this stage we were not concerned with exclusivity or inclusivity, but merely looking for 'features' which would contribute to our later, deeper analysis. Examples of mathematical behaviour classes were "defining", "into algebra", "using materials". Verbal behaviours are illustrated by labels such as "working out loud (together)", "revealing errors", "verbalising for approval (frequently their own)". The first set of data was then viewed through this lens, while the original categories of (a) to (r) were applied to the second set of data.

Thereafter, each new set of data that was collected was analysed and used to confirm the existing classifications or to suggest new ones. Where new classes were suggested, all previous data were reviewed again in the light

of the possible new classification. This collection and analysis process continued until the categories seemed to be stable. It is interesting to note that the categories (a) to (r), with the exception of the re-naming of (r), remained constant and inclusive, in terms of covering all episodes of discussion, throughout the analysis of the rest of the data. A description and the arguments for the methodology outlined above are dealt with in detail in Pirie (1991)

<u>'Pupil As Teacher'</u>

The rest of this paper is concerned with the data which formed one of the stable sub-categories, in terms of the pupils' verbal behaviour, that which we called 'Pupil as Teacher'

When a grouping of episodes where one of the pupils appeared to take on a role as teacher in the discussion began to materialise, we needed to explore more precisely what constituted the features of verbal teacher behaviour as distinct from pupil behaviour. Sinclair and Coulthard (1975) offered the following four main functions of teacher talk: informing, directing, eliciting and checking. In all our examples of talk within the 'Pupil as Teacher' category, one pupil exhibits most if not all of these verbal functions, but on closer examination it became obvious that we needed to distinguish between the pupil who spontaneously adopted the role of 'teacher', and the pupil who was cast in that role and used as a 'teacher' by another pupil. This distinction was evident in the verbal activity which initiated the particular episode of discussion. It was then appropriate to consider how faithfully each pupil adhered to the assigned roles and we further classified the behaviours as 'typical' or 'atypical'.

At this point it would seem helpful to offer examples of protocols which illustrate the types of talk we have been describing. We make no apology for the fact that we intend to do this through constructed , and not real examples. The following have been carefully based on actual pupil talk, but simplified in order to convey to the reader the specific features we wish

to highlight. 'T' denotes the pupil who is cast in the role of 'teacher' and 'P' another pupil.

First we illustrate our language categories. We have further categorised the discussion by noting which of the pupils is using which type of language. The following three excerpts are from pupils working on parallel lines and related angles.

(1) No Mathematical Language

P: Hey, Steve can you help me on this one please?.
T: O.K. Which one is it?
P: This one here
T: Well you see that's down from that, that's 70, so that's gonna be 70 as well (informing) , do you get that? (checking)
P: Um, I think so
T: Well, what do you think this one will be then?(eliciting)
P: Will it be 50?
T: Yeah, that's it, good!

(2) One Pupil Using Mathematical Language (in this case the 'teacher')

T: So, do you understand how these ones work? (checking)
P: Not really, how do I work out what this one will be?
T: Well, look how these lines are marked in the diagram (directing)
P: Mmm
T: Well, that tells you that they are parallel lines
P: Mmm
T: We know when a third line cuts across the parallel lines, it makes an F shape, these angles will be equal,they are corresponding angles (informing)
P: So it's 57 like that one?
T: Yes, that's right

(3) Both Pupils Using Mathematical Language

T: You look as if you're stuck
P: How is this supposed to be a corresponding angle?
T: Well, look how these lines are marked in the diagram (directing)
P: Yes
T: Well, that tells you that they're parallel lines
P: Oh, so if these are parallel, these angles will be the same, I remember
T: Yes, because when a third line cuts across the parallel lines, to make an F shape (informing), they're....(eliciting)
P: ...Corresponding angles!
T: Yes, that's it, good. And these angles will be equal so...(eliciting)

P: ...That's 57.
T: Yes, that's right.

Secondly we illustrate our categories of statements and the roles adopted by pupils.

(4) Abstractive Statements: Typical Teacher And Typical Pupil

Pupils here are working on sequences of patterns and the question "What patterns do you see in the of dots needed for each model in the sequence?"
T: Have you seen any pattern yet?(checking)
P: Well, I've done it for up to the sixth one and I've got the answers 1, 4, 9, 16, 25 and 36, but I can't really see a pattern
T: Well, look at the fourth one (directing), four dots across, four dots down and the answer is 16 dots. Four times four is(eliciting)
P: 16!
T: Yes, and for the fifth one? (eliciting)
P: 25
T: Yes, so they are all square numbers aren't they? They're called square numbers cos if you look at the models, they are all actual squares aren't they?(informing)
P: Oh yeah, So the tenth one would be, um, 100!
T: Yes.

(5) Operational Statements: Atypical Teacher and Atypical Pupil

In this excerpt pupils are plotting a graph.
P: Oh it's stupid! How do you know the axis to plot the 'yes' and 'no' on?
T: Let me see. (directing) Brainy here will explain
P: Go on then clever clogs
T: No, look at the example (directing), see they've put them along the bottom (informing), so it's obvious isn't it?
P: No it's not obvious, but I think I get it
T: So where do you think the numbers go then, thicky? (eliciting)
P: On the horizontal axis?
T: NO!

It should be mentioned here that we also had examples of excerpts in which one pupil was playing a 'typical' role and the other not.

One further observation made on each episode of 'Pupil as Teacher' discussion was whether the outcome was a success in terms of short term learning for the 'pupil'. Example 5 above illustrates a situation where the outcome was considered unsuccessful, in contrast with example 4.

It is important to emphasise here that we are working within a case study framework and not in any way in a position to make sweeping generalisations from our data. Nonetheless, some striking features have emerged from this detailed examination of the category of 'Pupil as Teacher' discussion.

Whenever a pupil elected to spontaneously adopt the role of 'teacher', the 'pupil' behaved in a 'typical' fashion, the outcome was successful, and in all but one case the 'teacher' behaved typically, used mathematical language and abstractive statements were made. We could hypothesise perhaps that pupils who choose the 'teacher's' role see it as one in which mathematical language must be used and the discussion cannot all be at the operational level. In contrast, when forced into the role of 'teacher', the majority of the 'teacher' pupils do not use mathematical language nor do the discussions contain abstractive statements. In more than one third of the cases, although the 'teacher' pupil displays the defining functions of a teacher, atypical behaviour is also evident.

Space does not permit the inclusion of further transcripts here. These will be available at PME and analysed and discussed during our presentation.

Bibliography

Barnes, D, Britton, J and Torbe, M. (eds.) (1986), Language, The Learner And The School (Third Edition, Penguin, England).

Barnes, D and Todd, F (1981), 'Talk in small learning groups: Analysis of strategies', in C. Adelman (ed.), Uttering and Muttering (GrantMcIntyre)

Glaser, B.G. and Strauss, A.L. (1968), The Discovery Of Grounded Theory (Weidenfeld and Nicolson, London).

Mehan, H (1985), 'Structure of classroom discourse', in T.A. Van Dijk (ed.), Handbook Of Discourse Analysis, Volume 3 (Academic Press).

Pirie, S.E.B. and Schwarzenberger R.L.E. (1988a), 'Mathematical Discussion and Mathematical Understanding', Educational Studies in Mathematics, vol. 19, pp. 459-470.

Pirie, S.E.B. and Schwarzenberger R.L.E. (1988b), 'Mathematical Discussion -It's What You Say and What You Do', Mathematics Teaching, vol.123,pp. 58-60.

Pirie, S.E.B. (1991), "Enumerative induction abstracts by generalisation, whereas analytic induction generalises by abstracting.", invited paper for the Conference on Qualitative Research Methodologies, Montana State University, Montana, U.S.A.

Sinclair, J.McH. and Coulthard, R.M. (1975), Towards An Analysis Of Discourse (Oxford University Press, London).

Webb, N.M. (1982), 'Peer interaction and learning in co-operative small groups', Journal of Educational Psychology, vol. 74, no. 5, pp642-55.

Znaniecki, F. (1968), The Method of Sociology, reprint of 1934 edition (Books Inc., New York).

TEACHER ATTITUDES AND INTERACTIONS IN COMPUTATIONAL ENVIRONMENTS

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This paper is a follow-up to that presented at PME XIV in Mexico. In it, we outline our findings of a three-year research project[1], in which we worked with a number of mathematics teachers on a year-long in-service course involving computational environments for mathematics classrooms. We map their attitudes towards (and interactions between) mathematics, mathematics teaching and computers.

Our previous work has indicated that teachers have a crucial role in helping children to learn within computational environments. We begin by clarifying two aspects of our position. First, we restrict our view of computational environments to a very small and rather special subset of computer software: namely applications which offer the learner — and in this case, the teacher — a medium in which to express his or her ideas about non-trivial mathematical domain(s). Logo is an example of what we intend to convey, although we see Logo as a paradigm for expressive computational media, and we do not view it as unique in adapting to this role (see Hoyles & Noss, 1991).

The second clarification we should make is that we believe that those involved in designing computational environments need to specify carefully the domain in which children might be expected to 'learn'. Our concern in this study was exclusively with the domain of mathematics. Thus we see the computer in general (and Logo in particular) as a catalyst for learning about mathematics. But mathematics is, par excellence, a conscious and reflective activity: it is, in our view, unreasonable to think that mathematical learning could be incidental or that it could occur outside of a carefully designed pedagogical framework.

Thus the teacher is crucial to the enterprise of learning mathematics — with or without a computer and it was, in large part, as a response to this belief that we turned our attention to teachers in setting up this study. Our aim was to attempt to map out some of the ways in which teachers thought and felt about employing the computer in their mathematics teaching, how their interactions with the computer influenced (and were influenced by) their pedagogical approach, and (to a lesser extent) how they integrated the computer into their classroom practice.

Methodologies of data-collection and analyses

The setting for the study was two intensive 30-day courses based in the University: each spanned an entire academic year. There were 13 participants in year 1 (1986/7) and 7 in year 2 (1987/8). All participants were secondary mathematics teachers, and many held positions of responsibility within their schools.

[1] The *Microworlds Project* : funded by the Economic and Social Research Council, 1986-9; Grant No. C00232364. Researchers were Celia Hoyles, Richard Noss and Rosamund Sutherland. The final report is in three volumes and is available from the authors. This paper is based on Volume 2 (Noss, Sutherland & Hoyles, 1990).

We were concerned to identify facets of the courses which seemed to interact with teacher attitudes and practices, but we were at least as interested in what the teachers actually did on the course and how teachers' existing views and attitudes influenced their activities; put another way, our focus of interest was on what teachers did *with* the course, as much as what the course did *to* them.

Our analysis is based on two levels: first, to try to elicit information about teachers' views and attitudes towards mathematics and its teaching — at the outset of the course and as it proceeded; second to gather data as to the ways in which teachers' professional skills influenced and were influenced by the course activities and the development of their ideas about mathematics, computation and its teaching. We also wanted to gain information concerning the ways in which the teachers perceived the transition to the classroom (and secondarily, *our* perceptions of this transition). Data consisted of semi-structured interviews, examination of teachers' projects, examination of teachers' reports of case studies of their pupils, observation notes of course activities, classroom observations and follow-up data obtained from post-course questionnaires.

Before we present an outline of our findings, it is helpful to offer an overview of the ways in which we analysed and subsequently presented our data in full. Essentially, we engaged in three levels of analysis: *case study*, *caricature*, and *cross-sectional.*

Case-studies were written of each teacher-participant from the raw data. Our final report presents three case studies of individual participants[1].

Caricatures of the course participants provided a synthesis of the views, attitudes and practices of a *cluster* of case studies: these attempt to draw attention to determinant characteristics and behaviours, and allow us to go beyond a simple presentation of detailed descriptions of disparate individuals.

Our caricatures are not *typifications*; they do not represent 'ideal types'[2] in the sense of being stripped of the detail which provides rich characterisation of attitude and behaviour; neither are they designed to be contradictory, they are not rivals to each other. The caricatures consist of attitudes and behaviours within a set of *dimensions* which coalesced as our data analysis proceeded, which when put together create a recognisable 'person'. The dimensions along which each caricature is described and distinguished are: view of mathematics; personal feelings of own mathematical competence; articulated aims of teaching mathematics; view of school mathematics itself and its relation to mathematics 'outside school'; expectations of pupils' mathematical attainment and attribution of success/failure; pedagogical orientation; attitude to computer use in mathematics; focus of course activities; classroom use of computers; and views of their own intervention strategies. Unavoidably, the dimensions reflect our own ideas about categories on which to gauge mathematics teaching and teachers. Thus the caricatures are *characterisations* of clusters of participants which allow us to 'look for' rather than 'look at' (Walker 1981), and hopefully, increase the generalisability of the data (Macdonald 1977).

[1]These were chosen because we were able to visit their classrooms considerably more often than others, and thus able to build a more comprehensive picture of their attitudes and classroom practices.
[2] As conceived by Weber, an ideal type is constructed by abstracting from elements which, although found in reality, are not present in this idealised form.

The *cross-sectional analysis* addresses issues which flow from examining the data 'horizontally', i.e. focussing on the dimensions and looking *across* the caricatures in order to reach more general conclusions concerning the interactions between teachers' attitudes and their activities on the course.

An outline of our conclusions

First, we introduce our caricatures: they are Mary — *the frustrated idealist*; Rowena — *the confident investigator*; Denis — *the controlling pragmatist*; Fiona — *the anxious traditionalist*; and Bob — *the curriculum deliverer*.

We begin with the observation that little discernible shift took place along the dimension which we have labelled 'view of mathematics'. This is perhaps not surprising as engendering such change is unlikely to happen even within a year. We might surmise that those teachers with strong mathematical backgrounds would be hardly likely to change their views of the subject as a result of the course, while those with weaker mathematical backgrounds were less likely to have explicitly articulated views of the subject in the first place.

There is a similar lack of any significant shift in terms of feelings of mathematical competence. However, we did find that *sharing* insecurity — or at least not concealing it — was a prerequisite for addressing the issue on a personal level, and at least a starting point for reevaluating one's feelings towards the subject. We also hypothesise that for any shift of this kind to take place, there needs to be some movement, some preparedness to change attitudes *before* any in-service course.

As far as the computer was concerned, the situation is somewhat different — there was a sharpening of participants' perception of the computer's role, possibly because views about computers were less entrenched than those about mathematics. To some extent, we would be most disappointed if this kind of shift had not taken place; but in saying that, we ignore the anecdotal evidence of many shorter, more technically-oriented courses, where such shifts do not occur and which sometimes engender shifts in the reverse direction. Thus, in terms of attitude to computer-use in mathematics, *all* the caricatures showed evidence of some shift. In so far as it is useful to generalise here, it seems that the prevalent mechanism was for the computer to offer a *means* to an end which had already been extant *before* the course — that is, it provided a vehicle for moving in a direction which had already been established. For example, Mary — somewhat insecure about mathematics, and 'computer illiterate' at the beginning — found in the computer work a means to *enjoy* mathematics: something she already *wanted* to do. Additionally, in beginning to see that she could enjoy and gain confidence in mathematics, she began to try to find ways in which she might make sense of this possibility for her pupils. Bob found a mechanism to allow him to satisfy what he perceived as the imperatives of the externally imposed pressure for curricular change emanating from recent developments in his school: the course thus offered a means to achieve an end that had already been externally set.

As far as the evidence of continued computer-use in school is concerned, the caricatures displayed a range of behaviour, from the integration of the computer as an organic part of pedagogical practice (Rowena and Bob) to an effective failure to use it all (Fiona), or to use it in strictly limited and controlled ways (Denis). We

therefore suggest that change in attitude to computer-use is therefore a necessary but not a sufficient condition to engender change in classroom practice.

It is certainly the case that there are distinct (though not disjoint) elements which come together to develop organic use of the computer. In order to disentangle them, we need to look at other dimensions. If, for example, we look at those caricatures which integrated the computer successfully into their practice (Rowena and Bob), we find two quite different mechanisms at work. Rowena had a flexible view of pupils and the curriculum, and adopted a proactive perspective on the question of curriculum change. Bob on the other hand, was much more curriculum-centred, and reactive towards change in general. The common element, however, was *confidence* in mathematics, mathematics teaching as well as with the computer itself. Thus, it seems that the view of mathematics and feelings towards it are critical, in that a sensitivity to mathematical ideas lends itself to using the computer in mathematically creative ways. That is, confidence is a prerequisite for constructive computer use. If it preexists, then we might reasonably expect teachers to incorporate the computer organically into their practice, as indeed we might if their experiences on the course were lengthy and deep enough. If not, then it is clearly difficult (though, as we have seen, not impossible) for a short course such as we offered to satisfy the requirement of generating confidence in mathematics and computing*and* to encourage constructive reflection on the computer's specific mathematical role.

We need to distinguish here between mathematical and computational confidence. While mathematical confidence existed initially among a number of course participants, the same did not generally exist in respect of the computer. Dealing with lack of confidence in mathematics is a difficult prospect, and the courses were only partially successful in dealing with it. Mary, for example, certainly did develop more confidence in mathematics, while Fiona did not. We can only surmise on the reason for this: it seems likely that it is related to something quite intangible — a feeling or sensitivity to mathematics which (in Mary's case) was certainly present before the course, and which the course developed and gave form. She began the course with a clear preference for the *professional* over the *personal:* her priorities were to find ways of *teaching* more effectively. Yet during the course, she developed considerable confidence in exploring mathematical ideas with the computer, and generated deep *personal* involvement with a number of mathematical ideas and projects. The course provided Mary with the realisation of her intuitive ideas about mathematics — that is it *was* indeed enjoyable, and above all, that she *could* do it successfully. We believe that it was this personal expression which brought about the shift in her pedagogical views: the computer in a sense *liberated* her to see that *enjoyment* and the *process* of doing mathematics could be realised, not only for her, but for the pupils. Fiona, on the other hand, had a view of mathematical activity which was strongly bounded by the demands of the curriculum, and for which she did not apparently share any significant personal commitment; she had considerable constraints on her in terms of her position in the school; and she did not allow herself to develop mathematically.

A shift also occurred along the dimension we have labelled 'expectations of pupils...'. This finding is by no means consistent across the caricatures, and even where it occurs, seems to change for different reasons (and with different 'end-points'). However, all three caricatures who initially maintained an inflexible view of

pupils' abilities and attainment, softened their view during the course. For example, Bob — whose central interest was in curriculum content — initially maintained a somewhat dismissive view of pupils' potential mathematical understandings and went so far as to predict that only those pupils who were good at mathematics would be likely to be 'successful' with the computer. In contrast, he articulated a post-course view that the computer could catalyse 'good teaching ' which could, in turn, produce change in pupils' attainment patterns.

We do *not* attribute such shifts to some mystical power of the computer: on the contrary, during the courses we placed heavy emphasis on *reflection on pupils' understandings* (and to a lesser extent — feelings), and asked each participant to produce detailed case studies of pupils working with the computer. Thus the case study demanded detailed attention to individual pupils, and there is little doubt that for some participants — like Bob — this may have been the first time that they had the opportunity of being able (or willing) to do so. Nevertheless, it is clear that the computer's presence facilitated this shift, if only by allowing participants a window into their pupils' (and, perhaps own) thinking. Here, at least, we think we can point to some causal mechanism for shifts in attitude, rather than decontextualised and trite claims regarding the 'effect of the course'.

A related, and perhaps more surprising shift occured in the domain of intervention strategy. This is a particularly interesting area, as we think we can discern a relationship between changes in attitudes towards intervention and participants' main focus of course activities.

Let us take Mary as an example whose developing sensitivity to mathematics was discussed above. When we turn to Mary's view of intervention, we find that she became more reflective on what constituted an effective pedagogic strategy. She started with a *mimimalist* position (effectively one of non-intervention) but later *quantitatively* intervened more — precisely in order to bring to pupils' attention mathematical ideas and notions that perhaps she had previously not considered important or had simply overlooked. Thus Mary did shift her view of mathematics pedagogy somewhat towards the explication of mathematics content alongside the promotion of purely process/affect issues. Essentially, the course catalysed for her the *possibility* of synthesising process and content both personally and professionally.

Some participants effected *qualitative* change in their view of pedagogy. For example, Rowena began the course with a strongly held set of beliefs about mathematical investigations, centred around the primacy of *doing* above understanding: yet she was aware that she spent 'too much time' directing pupils' work. We suggest, not that she fundamentally reappraised her beliefs, but that she found, by working with the computer, a means of relieving the tensions which she explicitly felt existed between her theory and her practice: that is, she found a mode of intervention through which she could prioritise *both* her preference for an investigative approach *and* a focus on helping pupils to understand mathematical content. Thus we find a qualitative as well as quantitative (less) shift in her intervention strategies.

Rowena's emerging resolution of her dilemma was, we suggest, a result of her *personal* engagement on the course. She found through, for example, Logo programming, a means for *herself* to have the opportunity to engage with mathematics in the same ways she prefers for her pupils. She saw at the same time

that simple engagement with computer activities was not sufficient to count as mathematics, and she began to synthesise *doing* with *understanding*. She found by engaging within an explicit pedagogical framework provided by the course, that she could reflect on and reappraise her view of mathematical ideas *without* conflicting with her strongly held beliefs about mathematics and its teaching. It was this *personal* experience which led her to reflect upon her pedagogical activities.

In contrast, an expressed preference for course activities closely linked to pupils' needs (at an extreme, to the total exclusion of any exploratory activities for their own sake) seemed often to be a manifestation of an anxiety about mathematics and the teacher's own mathematical competence (we are not, of course, suggesting that only anxious teachers express interest in their pupils — only that we noticed a reverse tendency). Fiona, for example, never really became personally (as opposed to professionally) involved with the computational activities of the course, and produced an extended project which was directly related to and constrained by the school curriculum. Fiona was a practitioner of limited experience who had not hitherto reflected deeply on the theoretical basis which implicitly informed her practice. While we do not question Fiona's commitment to her children, we do not think that this commitment grew out of any *explicit* set of beliefs about the ways they learn, or need to learn, mathematics. It is here that we can locate some influence of the course.

We would want to characterise Fiona's developing reflective attitude towards classroom practice as one of commitment, but still on a pragmatic rather than an organic level. That is, while we have characterised a change in Fiona's thinking, we interpret this as primarily arising from her anxieties concerning externally imposed changes in the curriculum, as well as the introduction of the computer which had brought about the need for change — perhaps almost for survival.

We have used the terms *reactive* and *proactive* to describe the motivations with which participants came to the course. By this crude classification, we would have to describe Fiona and Denis as reactive. Yet the complexity hidden in this simple polarisation is evident if we compare Fiona to Denis. Denis needed, both personally and professionally, to be in control. He saw the curriculum innovation posed by the computer as a potential source of disruption to this control — a threat even. Denis had a model of student learning which was both implicit and authoritarian — and we have reason to believe that he viewed his *own* learning in similar terms. He was not confident, either in his mathematical or computational ideas and expressed the view that he had to be "100% au fait" with any new idea before he could think of introducing it into the classroom; in the absence of such a level of competence and confidence, the potential for disruption of his control of events was simply too great to risk.

Nevertheless, in one respect, Denis's attitudes were influenced by the course in that he, perhaps for the first time, became aware of misunderstandings of his pupils and their difficulties with mathematics. He saw these as being amenable —with or without the computer — to a similarly authoritarian solution: that is, he was able to incorporate this knowledge into his existing framework and adopt remediation techniques based on his existing approach: he even encouraged other teachers in his department to similarly use the computer. To sum up, the essential distinction

between Fiona and Bob hinged on the degree of reflection each brought to bear on their activities.

Concluding remarks

We conclude with a few remarks on the dynamics of change we set out to investigate. In reading this paper, it is perhaps easy to form the impression that we set out to *push* teachers in new directions, to conform to a new set of established goals. While it would be attractive to suggest that this is not what we attempted to do, we did, of course, present on the courses a particular view of pedagogy and mathematical education. These are reflected in the framework underpinning the caricatures, and in the dimensions along which attitude change was identified. It is difficult to escape from the reality that some participants (but by no means all) did see us as authoritative figures to whom — even when they disagreed with us — they should defer.

Schools are usefully constructed as settings with their own discourses: teachers are not so much constrained by their setting as reciprocally acting on it and by it. We therefore need to be realistic about the explanations for participants' changes in emphasis, attitude and practice — many of these have little or anything to do with the course. The field of computing is replete with constraints such as difficulties of gaining access to sufficient computers (and computing power), the technical orientation of the laboratory, the current necessity for children to go to computers rather than the computers to go to the children. Some of these constraints will disappear in the course of time; others will not. It is equally clear that for the forseeable future, the most productive application of the computer within the context of mathematical education, will continue to be in terms of the kind of expressive medium we attempted to develop on our courses. If this is the case, *the* crucial issue will continue to be the research and development of ways to integrate the computer into teachers' practice, and to find ways in which teachers' practices may come to be reflected in the medium.

References

MACDONALD B. (1977) The portrayal of persons as evaluation data. In: NORRIS N. (ed.) *SAFARI Theory in Practice Papers Two*. Occasional Publications 4, CARE pp. 50 - 67. University of East Anglia.

HOYLES C. & NOSS R. (Eds.) (1991) *Learning Mathematics and Logo*. Cambridge: MIT Press.

NOSS R., SUTHERLAND R., & HOYLES C. (1990) *Teacher attitudes and interactions*. Report of the *Microworlds* Project, Volume 2. Institute of Education, University of London. [See also: SUTHERLAND R., HOYLES C. AND NOSS R. (1990)*The Microworlds Course*: Report of the *Microworlds* Project, Volume 1; HOYLES C., NOSS R. AND SUTHERLAND R. (1990) *The Ratio and Proportion Microworld*. Report of the *Microworlds* Project, Volume 3]

WALKER R. (1981) On Fiction in Educational Research: and I don't mean Cyril Burt. In: SMETHERHAM D. (ed.): *Practicing Evaluation*.. Nafferton.

CHILDREN'S UNDERSTANDING OF MEASUREMENT

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Cultural influences on 6 to 8 year-old children's measurement ability was investigated in situations in which they could/could not use rulers. Children worked in pairs to solve tasks that required comparing the length of two lines. The use of rulers was not an overlearned procedure at this age level and conventions had to be established for similar procedures to be adopted. Measuring with non-conventional instruments (strings) was more difficult because it required re-invention of concepts embedded in rulers such as iteration and subdivision of units. Children's difficulties in using a broken ruler indicate that measurement procedures may be used but not fully understood at this age level.

Measurement has been analyzed by researchers working within two different traditions. The first of these draws on the work of Piaget. Piagetian researchers analyzed some of the invariants central to the understanding of measurement, namely transitive inferences and the idea of unit. In this research tradition, there is no specific interest in measurement systems available in the culture but only in the logico-mathematical invariants which underlie quantification. Accordingly, Piagetian studies are typically planned so as to exclude from the experimental situations cultural instruments which children could use in measuring. Children are asked to reason about lines of unknown lengths or to measure without relying on cultural technologies (see Piaget, Inhelder & Szeminska, 1960). They have to re-invent measurement concepts not being able to draw on instruments in which these concepts are embedded.

The second research tradition has been influenced by the ideas of Vygotsky and Luria who emphasized the socio-historical nature of human thinking. Studies within this tradition have brought culturally developed systems of quantification into the analysis of measurement. These researchers analyzed children's and adults' reasoning about logico-mathematical invariants and the process of quantification within specific measurement

systems. Gal'perin and Georgiev (1969), for example, investigated how children dealt with the idea that the same number might indicate different quantities if different units (such as tablespoons and teaspoons) are used in measurement. Carraher (1985) posed the same type of question in a study of children's ability to compare amounts of money when the number of coins was the same but different denominations were used. Saxe and Moylan (1982) also studied the interplay between logico-mathematical concepts and quantification within specific cultural systems by looking at inference-making among children and adults in Papua New Guinea, where indigenous systems of measurement for the length of bags use body parts, units which vary in size across persons.

Despite their interesting contributions, these studies have not allowed for a clear analysis of what cultural measurement systems contribute to people's measurement abilities because they have not included comparisons between situations in which conventional systems are available with others in which they are not available. Piagetian studies on measurement required children to re-invent measurement procedures using non-conventional resources. In contrast, studies which have made these technologies available to their subjects have allowed them ample access to the technologies without testing for the understanding of some of the principles of quantification already provided by the measurement technology. For example, when one uses a ruler to measure length, iteration and subdivision of units are given on the ruler for the benefit of its users. However, there is little information on whether children benefit from using such ready-made systems and accomplish their understanding in context before they could re-invent these concepts. The concepts may be more difficult to re-invent than

o use when they are available in the culture. On the other hand, understanding may involve essentially the same process as inventing (see Piaget, 1969).

In this study, children were asked to carry out measurement under different conditions which either included or excluded cultural technologies. In order to investigate the existence of procedural solutions without understanding, a third condition was created, in which a constant error of measurement was introduced so that children could not simply "read" the numbers from their rulers.

Method

Design. The task was carried out by children working in pairs in one of three conditions. In the String Condition, children had identical strings to measure with. In the Ruler Condition, they were given rulers marked in centimetres and half-centimetres on one side and centimetres and millimetres on the other. In the Broken Ruler Condition, one of the children in the pair had a broken ruler starting at four centimetres, a circumstance clearly pointed out to the children. Comparisons between the String Condition and the Ruler Condition test whether the availability of a ruler significantly improves children's performance in this task. Comparisons between the Ruler and the Broken Ruler condition test whether children simply obtain readings from the ruler and make decisions on the basis of the numbers read without understanding what the numbers on the ruler mean. Simple reading of the numbers on the broken ruler gave wrong measures and often led to wrong conclusions.

Children were sampled from two grade levels in two schools in Oxford and randomly assigned to one of the three experimental conditions. Teachers were then asked to form same-sex pairs of children within the list for each condition. Fifteen pairs of

children were tested in each condition.

Procedure. Children (6 to 8 years old) worked in pairs. Each member of the pair was placed in a different room so that they could neither see nor talk to each other directly. They spoke over telephones connected to a tape-recorder which allowed full transcripts to be obtained. Transcripts were later coordinated with the experimenter's notes for data analysis.

The children were told that they were going to play 'a mental game, like those played in Crystal Maze' (a TV show well known to British children at this age level). This instruction helped establish an atmosphere in the experimental situation requiring reasoning and cooperation by exchanging information. The physical separation with the telephone connection also meant that the task had some game-like purpose.

Each child in the pair received a sheet with a line on it. Their task was to find out whether the line on one child's sheet was longer, shorter or equal to the line on the other child's sheet. Children were asked to give a mutually agreed response to each trial. The analysis of performance across situations was based on this response obtained from the pair. If disagreements emerged, children were asked to continue talking until they reached consensus.

The lengths of the lines in the task were chosen in order to test for knowledge of the different logico-mathematical invariants of measurement. The structure of the task is described in Table 1, where types of item are presented.

The first trial was a practice trial. After responding, the children came together and compared their lines thus obtaining direct feedback. When they had come to a wrong answer on this item, they were encouraged to show each other how they had measured. The subsequent trials were grouped in two blocks

...ich contained one item of each of the three types described above. Order of presentation was systematically varied across children within blocks, with three orders being used (ABC; BCA; CAB). Feedback was given after each of the three trials in the first block. Feedback for the last three trials was deferred until the end of the session.

Table 1

Let the lines be A and B and one unit in terms of string-length be S. In all cases, S=7 cm. For any case, if B is greater/smaller than A, the difference is of 1 cm.

Practice item
A=S; B=S

Transitive inference only items
1) A=S; B>S　　2) A=S; B<S

Iteration of units items
1) A=2S; B>2S　　2) A=2S; B<2S

Subdivision of units items
1) A=.5S; B>.5S　　2) A=.5S; B<.5S

Results

The number of correct responses given by each pair of children was used to ascertain the effect of condition upon performance. The mean number of correct responses by condition is presented in Table 2. An ANOVA with the total number of correct responses as dependent variable showed a significant effect of condition ($F = 4.47$; $p = .017$). The significance of the differences between the means was tested by the Newman-Keuls method, which showed the difference between the String and the Ruler conditions to be significant ($Q = 4.104$, $p<.05$). The other two comparisons did not produce significant differences.

Table 2

Condition	Mean number of correct responses (n of trials = 6)
String	3.69
Broken Ruler	4.13
Ruler	5.50

Three further ANOVA's were carried out in which the three types of item were considered separately. All three analyses showed a significant main effect of condition (for the transitive inference items, $F = 3.561$, $p <.05$; for the iteration of units items, $F = 6.754$; $p<.01$; for the subdivision of unit items, $F = 3.362$; $p<.05$).

The most salient aspect of the results however was the qualitative difference in the tasks the children faced under the different conditions. Children in the String condition had to <u>devise a method</u> for measuring with the string. Using the string iteratively was not difficult. If one child came up with the idea, he/she was usually able to instruct their partner over the telephone. However, procedures for obtaining subdivision were not observed often. Children tended to estimate whether or not the lines were equal to half the length of the string. Children in the Ruler condition did not have to devise such procedures, but simply make sure that they were starting at the same point (0, 1 or the edge of the ruler) and that they were referring to the same units (centimetres or half centimetres). <u>Establishing conventions</u> was their main problem. Finally, children in the Broken Ruler condition had to <u>both establish conventions and devise a procedure to avoid the constant error</u> introduced by the broken ruler. This condition induced a lot of discussion on how to deal with the constant error and produced a deeper analysis of measurement than either of the other two conditions.

Discussion

Children did better measuring with the support of the cultural instrument than without it. However, children may profit from a technology even without fully understanding it, as performance in the Broken Ruler condition indicates. These

results have interesting educational implications. First, children need not reinvent mathematical solutions anew in order to learn them. They can profit from their use and draw on them as tools. Second, they may behave as if they understand these solutions even without fully understanding them. Working under conditions which require new ways of using the measurement technology seems to be both a way of probing into understanding and a rich educational experience.

At a more general level, the study also demonstrated the importance of the particular type of cooperation created here in bringing children to represent non-verbal procedures verbally. In order to direct their partners to obtain measures in a comparable fashion, children accomplished levels of explicitness which were quite surprising for these age levels. If verbal representation can bring children to higher levels of awareness of their logico-mathematical knowledge, this situation may offer not just a good research setting but also a profitable educational situation.

References

Carraher, T. N. (1985). The decimal system. Understanding and notation. In L. Streefland (Ed.), Proceedings of the Ninth International Conference for the Psychology of Mathematics Education. Vol.1. Utrecht: University of Utrecht Press.

Gal'perin, P. Ya. & Georgiev, L. S. (1969). The formation of elementary mathematical notions. In J. Kilpatrick (Ed.), Soviet Studies in the Psychology of Learning and Teaching Mathematics. Vol. 1, The learning of mathematical concepts. Chicago: University of Chicago Press.

Piaget, J. (1969). Science of Education and the Psychology of the Child. New York: Viking Press.

Piaget, J., Inhelder, B. & Szeminska, A. (1960). The Child's Conception of Geometry. New York: Harper and Row.

Saxe, G. B. & Moylan, T. (1982). The development of measurement operations among the Oksapmin of Papua New Guinea. Child Development, 53, 1242-1248.

A 3-DIMENSION CONCEPTUAL SPACE OF TRANSFORMATIONS FOR THE STUDY OF THE INTUITION OF INFINITY IN PLANE GEOMETRY

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Summary

Since the dawn of civilizations the concept of infinity has played an important role in almost every branch of human knowledge, and in mathematics its study has presented many difficulties and disputes. In education, many problems have been reported concerning learning of concepts related to infinity. Such an elusive and important concept of human mental activity should represent an interesting subject for cognitive psychology. Paradoxically, none of the different theoretical approaches of cognitive psychology has studied this concept deeply enough. We believe that different concepts of infinity, associated to different qualities and features, activate different cognitive processes, and because of this they should be studied systematically. The main objective of the present paper is to propose some clues to the systematization of the study of infinity, particularly in what concerns intuitive aspects in plane geometry. A 3-dimension conceptual space is presented, where the dimensions are: *Type* of infinity: whether it is conceived after a divergent or convergent operation; *Nature of the content* of the operation: cardinal or spatial; and the *Coordinations* of one or two attributes varying in type and/or nature. With such a space it could be possible to observe, throughout children's development, the operational profile (with their performances), and the epistemological difficulties of some important factors related to infinity. Moreover it could help the elaboration of new concepts for the understanding of infinity and its educational consequences.

The Problem

Throughout mankind's history, the concept of infinity has played an important role in almost every branch of human knowledge, fascinating and thrilling philosophers, theologians, scientists and mathematicians. Besides, the concept of infinity is present in cultures which differ very much in social practices and in geographical and climatic conditions, and it seems to be something peculiar to human cognitive activity only.

In mathematics, its study has presented many difficulties and disputes. Already since the early time of Zeno and the classical Greek philosophers, infinity, full of counter-intuitive features, has always been an extremely controversial and elusive concept. In mathematics, Greek thinkers, the scholars of the XVIIth century -who coined the tools of infinitesimal

calculus- and those of the XIXth century -including the revolutionary Georg Cantor- worked hard on this topic facing countless puzzles and paradoxes and developing intense polemics about this peculiar concept.
Such an important and particular concept of human mental activity should represent an interesting subject for cognitive psychology. This, mainly for two reasons:

a) because of the important role that this concept has played in the different disciplines of human knowledge, and
b) because it is a rich and representative concept of a dimension of mental activity not based on direct experience.

Paradoxically, none of the different theoretical approaches of cognitive psychology has studied this concept deeply enough (Fischbein, Tirosh & Hess, 1979; Núñez, in press); not even approaches close to mathematics, logic and formal disciplines in general, like the information-processing approach, or Piaget's genetic approach. Both approaches being close to formal disciplines because of the objects they intend to study and because of the conceptual tools they use to model findings.

For a clear delimitation of the extension of the concept of infinity, we decided to focus our study on the concept of infinity in mathematics. This, because the formal nature of the latter facilitates the operationalization of research designs and, even more important, because mathematics is a domain where infinity plays a very important role: "The concept of infinity lies at the core of mathematics" (Sondheimer & Rogerson, 1981, p. 86), "The use of infinity ... constitutes the profession of mathematics" (Zippin, 1962, p. 5), "David Hilbert ... defined mathematics as 'the science of infinity'" (Zippin, 1962, p. 3); "Mathematics, in one view, is the science of infinity" (Davis & Hersh, 1980, p. 152).

In cognitive psychology, literature in this area is, in general, oriented mainly towards pedagogical applications in disciplines where infinity is present, without searching a deeper understanding of the phenomena. In addition, efforts seem to be isolated, discontinuous, and poor in theoretical ties. Besides, there is a lack of precision: under the name "infinity" is confined sometimes potential and sometimes actual infinities; huge and small infinities; referred to quantity and to spaces; in mathematics, related to different contexts like series, geometry, limits;

under static or dynamic conceptions; etc. In order to deal with this difficulties and to develop a solid theoretical corpus that permits the creation of new concepts for a better discrimination of the phenomena that now aren't well identified, we suggested to consider the cognitive activity that infinity in mathematics requires as an independent scientific object for cognitive psychology (Núñez, 1990).
Our hypothesis is that different qualities and features associated to infinity activate different cognitive processes, and because of this they should be studied ***systematically***. The main objective of the present paper is to propose some clues to the systematization of the study of infinity.

The conceptual space*

We believe that one way to study psycho-cognitive aspects of the concept of infinity as *pure* as possible (i.e., trying to isolate the mathematical training) is to focus research on intuitive aspects. By intuition we understand "direct, global, self-evident forms of knowledge" (Fischbein et al, 1979, p. 5). For this reason we have chosen the domain of plane geometry, in which it is possible to ask children about number and sizes of figures without recalling specific school-learned knowledge or technical notions.
In order to simplify and to operationalize our approach, we assume that a plane figure can be transformed (its attributes being operated iteratively) in terms of number (cardinality (#)), and/or in space ((S); vertically and/or horizontally, being enlarged (D), or diminished (C)). We suggest the study of at least the following qualities present in these iterative operations:

1) ***Type*** of infinity: whether it is conceived after a **divergent** ($T_{(D)}$) or **convergent** ($T_{(C)}$) operation
2) ***Nature of the content*** of the operation: whether it is **cardinal** ($N_{(\#)}$); infinite quantities) or **spatial** ($N_{(S)}$; infinite big or small spaces).

* An experimental application of this conceptual space is being done in our University, having children of 8, 10, 12 and 14 years old as subjects. Results are to be obtained in March 1991.

3) ***Coordinations*** of type & nature: $T_{(D)}N_{(\#)}$; $T_{(D)}N_{(S)}$; $T_{(C)}N_{(S)}$
4) ***Contextual*** and figurative features

In what concerns the first quality, ***type*** of infinity, and this from a developmental point of view, important evidence shows that the idea of convergence is mastered and understood much later than that of divergence (Piaget & Inhelder, 1948; Langford, 1974; Taback, 1975). Even if we analyze the history of the concept of infinity in mathematics we observe a similar tendency (Núñez, 1990). Related to the second quality, ***nature of the content***, we believe that a distinction between a cardinal content and a spatial content should be made, especially when talking about infinity. We share the opinion of D. Tall who says "cardinal infinity is ... only one of a choice of possible extensions of the number concept case. It is therefore inappropriate to judge the 'correctness' of intuitions of infinity within a cardinal framework alone, especially those intuitions which relate to measurement rather than one-one correspondence" (Tall, 1980, p. 271).

Qualities of an attribute iterated infinitely

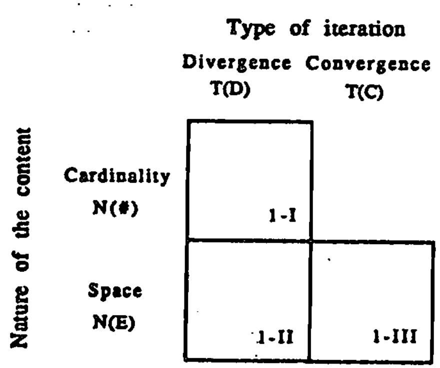

Fig. 1

Coming back to the space of transformations, a simple iterative operation (an attribute being iterated infinitely) could be seen as having one **type** and one kind of **nature**. So, as we can see in figure 1, it is possible to classify these operations in three main categories: divergent-cardinality $T_{(D)}N_{(\#)}$, divergent-space $T_{(D)}N_{(S)}$, and convergent-space $T_{(C)}N_{(S)}$ (convergent-cardinality doesn't exist).

Now, if we consider that other attributes may be iterated infinitely at the same time, we have a new interesting dimension to be studied: the **coordinations** of the qualities just mentioned. In our opinion, what is interesting is not only to observe how many attributes should be coordinated in a given situation, but to observe what are the attributes of these elements.

Figure 2 shows a classification of coordinations of the qualities represented in Fig. 1, when two attributes are iterated infinitely.

Coordination of two attributes iterated infinitely

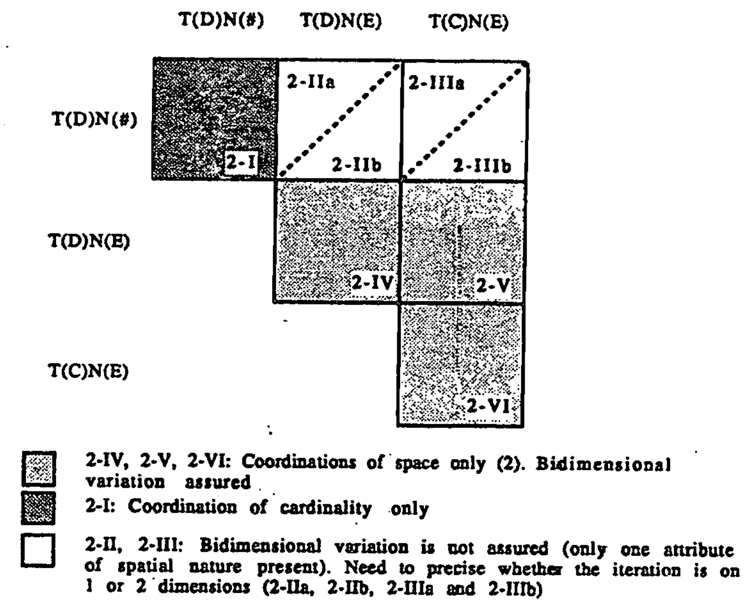

Fig. 2

Situations on the diagonal, 2-I, 2-IV and 2-VI are "symmetric" because they coordinate attributes of the same type and nature. Situations 2-IV, 2-V and 2-VI coordinate attributes with the same spatial nature. In these cases a bidimensional spatial variation is assured. Situation 2-I coordinates only attributes of cardinality. In cases 2-II and 2-III, a bidimensional spatial variation is not assured because there is only one attribute of spatial nature present, the second attribute being of cardinal nature. In these cases, we must precise whether the spatial iteration is

on 1 or 2 dimensions, simultaneously with the cardinal iteration. So we have 2-IIa, 2-IIIa for 1 dimension, and 2-IIb, 2-IIIb for two dimensions. Finally the last quality we consider necessary to study is the influence of the context and figurative aspects, but we're not going to treat this point in this report.

These different qualities of the operations related to infinity could be represented in a 3-dimensional space, a space of transformations (Fig. 3), where axes are each of the most simple qualities that an attribute infinitely iterated may have ($T_{(D)}N_{(\#)}$; $T_{(D)}N_{(S)}$; $T_{(C)}N_{(S)}$).

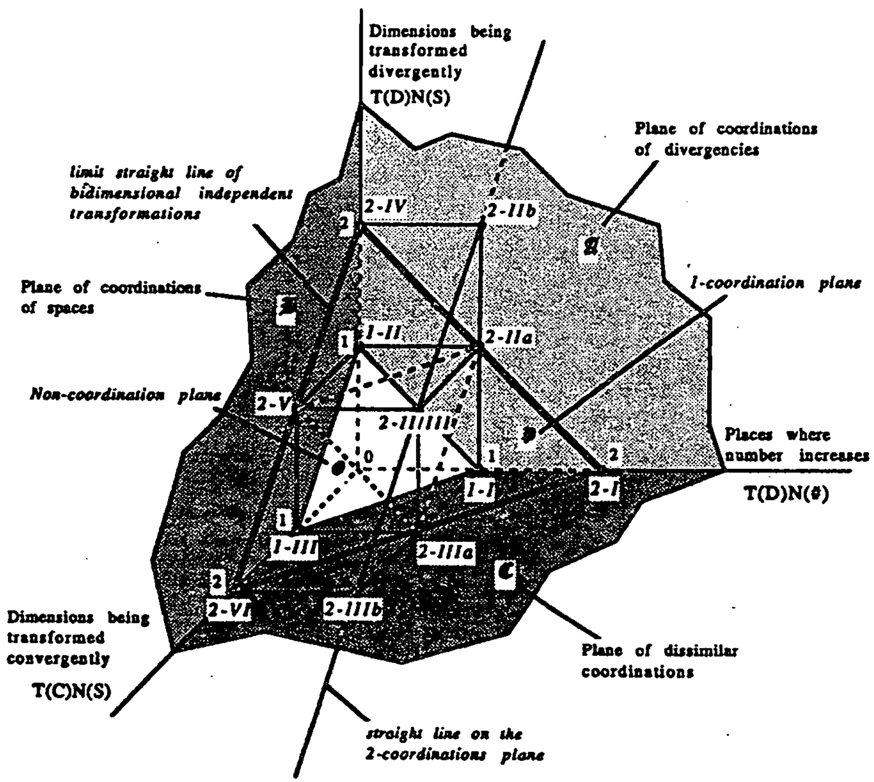

Fig. 3

Thus, basic features of an infinitely iterative process of a plane object are well identified as to be studied experimentally. Points in the space are each of the combinations of coordinations mentioned above, from 1-I

(Fig. 1) to 2-VI (Fig. 2) which hypothetically activate different processes from a cognitive point of view. Notice that a 12th point appears, 2-II/III which represents the coordination of attributes being iterated infinitely both in cardinal and spatial nature, and in the latter, one spatial dimension being convergent and the other divergent.

We believe that such a model facilitates the conception and the operationalization of our hypothesis. For instance, among other hypothesis we could mention that, we postulate that in general the closer one gets to the origin (no transformation), the easier to deal with the transformation; that situations on axis $T_{(C)}N_{(S)}$ are more difficult than those on axis $T_{(D)}N_{(S)}$ (due to the differences of difficulty levels between divergence and convergence mentioned above); that situations related to infinity represented by points on plane $\mathfrak{C}$ are much more difficult to deal with, than those on $\mathfrak{A}$ or $\mathfrak{B}$ (because there is a coordination of qualities different in type and nature, which demands an activity of a superior level than that demanded in plane $\mathfrak{A}$ or $\mathfrak{B}$, where at least qualities of the same order are present, type (divergence) and nature (space) respectively); etc.

Conclusions

Since very little is known about the cognitive activity related to the concept of infinity, a systematization of its study is imperative. We believe that an attempt of identification (and isolation) of pertinent factors is a subject of scientific interest and importance, which could be materialized with a conceptual space of transformations like the one presented. With such a space it could be possible to *visualize*, throughout children's development, the operational profile (with their performances), and the epistemological difficulties of some important factors related to infinity. And this, by having a view of the whole space, that is, considering pertinent features simultaneously. Besides, the fundamental aspects of cognitive activity that is possible to study with this approach, could bring important applications for education in mathematics and disciplines related to abstract reasoning.

References:

Davis P. J. & Hersh, R. (1980). *The mathematical experience*. Boston: Birkhäuser.

Fischbein, E., Tirosh, D. & Hess, P. (1979). The Intuition of Infinity. *Educational Studies in Mathematics, 10*, 3-40.

Langford, P.E. (1974). Development of concepts of infinity and limit in mathematics. *Archives de Psychologie, 42* (167-168), 311-322.

Núñez, R. (1990). Infinity in Mathematics as a scientific subject for cognitive psychology. *Proceedings of the 14th International Conference Psychology of Mathematics Education, (Oaxtepec, Mexico)*, Vol. I, pp. 77-84.

Núñez, R. (in press). La psychologie cognitive et le concept d'infini en mathématiques: une perspective historique. *Note di Matematica e Fisica.*

Piaget, J. & Inhelder, B. (1948). *La représentation de l'espace chez l'enfant*. Paris: P.U.F.

Sondheimer, E. & Rogerson, A. (1981). *Numbers and Infinity. A historical account of mathematical concepts*. Cambridge: University Press.

Taback, S. (1975). The child's concept of limit. In Rosskopf, M. F. (Ed.), *Six Piagetian studies in mathematics education.* Teachers College Press.

Tall, D. O. (1980). The notion of infinite measuring number and its relevance in the intuition of infinity. *Educational Studies in Mathematics, 11*, 271-284.

Zippin, L. (1962). *Uses of Infinity*. New York: Random House.

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THE STATUS OF CHILDREN'S CONSTRUCTION OF RELATIONSHIPS

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SUMMARY

American children entering sixth grade (average age 11: 5) were individually interviewed to determine their understanding of the relationships involved in inferring the color of various boxes in visual patterns of the form 2n, 4n and 4n + 2. The tasks were very difficult for the children. Many children responded, "I can't tell", even for 103 in an odd-even pattern. Many children extended the patterns physically or mentally. Few applied knowledge of 2n or 4n or 4n + 2 relationships.

This study is ongoing; data are being gathered from comparably aged children in British schools.

INTRODUCTION

"What is mathematics?" George Polya was once asked. "Mathematics is being lazy," he said. "Mathematics is letting the principles and the relationships do the work for you so that you don't have to do the work yourself." (O'Brien, 1986)

This research is concerned with children's construction of relationships. It arose in the context of an undergraduate math education class where students were asked to interview youngsters on tasks devised by the author (O'Brien, 1977) in order to learn about children's mathematical thinking. The data gathered by the pre-service teachers were so surprising that the author decided to pursue the issue more systematically.

This work is part of an ongoing project. Data from American students have been gathered and analyzed. Data from British schoolchildren are being gathered at present.

TASKS, METHOD, SAMPLE

The tasks were as follows:

"Look at the following patterns. In each case you will be asked about blocks which are showing and blocks which are not shown. Please answer my questions and please explain how you reached your answer."

In each case, children were asked about the color of various boxes which were visually apparent (such as Box 6) in order to check that they understood the task. Then they were asked about boxes not shown. Inference -- the deriving of information which is not perceptually available -- was the focus of this research.

The tasks were as follows:

Pattern 1:

1 2 3 4 5 6 7 8 9 10 11 12 13

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What is the color of ...

Box 18? Box 46? Box 103?

Pattern 2:

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15

What is the color of ....

Box 21? Box 40? Box 1047?

Pattern 3:

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15

What is the color of ...

Box 21? Box 40? Box 1047?

Individual interviews of 30 fifth grade (age 10 : 11 - 12 : 2; mean 11 : 5) children were conducted by a twenty-year veteran school teacher under the direction of the author. The children were pupils in a rural midwestern American public (i.e. state supported) school. The interviews took place in late summer, just before the children returned to school for sixth grade. The mean percentile group standardized-test scores (National Test of Basic Skills, 1985), from the children's school records, were: Computation: 81.4; Applications: 66.8; Concepts: 73.3.

RESULTS

The results were initially categorized as follows:

1. "Don't know". Child remains silent or answers a color with no justification ("Black". Why? [Silence])

2. Physical extension. The child draws an extension of the pattern (draws the blocks and shades them in) or uses some other written method of representing or extending the pattern.

3. Mental extension. The pupil mentally counts on (by ones, twos or fours) from the last block shown to the block in question. There is no written representation. (Children were often observed to count aloud or silently or to move rhythmically.)

4. The use of relationships (other than multiplication/division, which is Category 5).

All responses were classified with respect to their justification, not to the correctness of the child's black/white answer.

Some justifications were mathematically correct, as in "1047 [Pattern 2] is white because all the blacks are even."

Some justifications were incorrect, as in [Pattern 1] "40 is black because 10 is white and the tens [i.e., 10, 20, 30, 40] go white, black, white, black." Also, [Pattern 1] "18 is white because 9 is white".

Some justifications were indeterminate, as in [Pattern 2] "40 is white because 4 is white." (Does the child think that all multiples of 4 are white? That all even numbers are white? That anything with a digit "4" is white?)

5. Multiplication/division relationship (including odd/even for Pattern 1). ("1047 is black [Pattern 2] because it's not divisible by 4.")

The data (per cent of all responses) were as follows:

	PATTERN								
	1			2			3		
Category	18	46	103	21	40	1047	21	40	1047
1. Don't Know	3	17	40	0	27	70	3	23	73
2. Physical	60	13	0	60	17	0	77	13	0
3. Mental	13	17	0	26	17	0	20	20	0
4. Relation	10	27	30	3	20	7	0	43	27
5. Multiply/ Divide	13	27	30	10	20	23	0	0	0

SUMMARY OF RESULTS

The children in this study were not far along in being mathematicians in Polya's sense. That is, they were not highly advanced in "letting the principles and the relationships do the work so that you don't have to do the work yourself."

The most striking finding was the high per cent of responses in the "Don't Know" category. For example, 40 per cent of children at age 11 fit this category for 103 in Pattern 1 and 70 per cent said "Don't know" for 1047 in Pattern 2.

Striking also was the apparent need for pupils to extend the pattern from the last box visible. For the smallest number in each pattern (i.e, 18, 21, and 21), 73 per cent, 86 per cent, and 97 per cent of children extended the pattern physically or mentally.

Finally, the low per cent of responses in category 5 is worth noting. The patterns were, respectively, 2n, 4n, and 4n + 2. Only 30 per cent employed odd/even (or "multiple of 2") for 103 in Pattern 1, only 23 per cent mentioned "divisible by 4" (or "multiple of 4") for 1047 in Pattern 2, and no one saw the 4n + 2 in Pattern 3.

DISCUSSION

The findings here match up with the data gathered by the undergraduates before the present study was undertaken.

Still, this is a preliminary study and it leaves a great deal to be explored.

1. Obviously, the relational justifications (Categories 4 and 5) need further exploration. What, for instance, is the status and evolution of the logical necessity of the justifications? How do they grow out of (if they do) the physical and mental extensions?

2. It is not known whether the low level of relational thinking is the result of a) some developmental limitation, b) an anomalous sample, c) a mathematics curriculum in which relationships received little emphasis, or d) some other reason. If the data gathered from the (comparably aged) British sample significantly exceed those here in the quality of relational thinking, it can be said that there is no developmental limitation on children's relational thinking as assessed by the present tasks.

3. As with previous research by the writer (O'Brien and Casey, 1983 a and b) where 75 per cent or more of American fourth and fifth graders could not construct a multiplicative "story problem" or real-life situation for the stimulus 6 X 3 = 18 (despite the fact that all children could produce that number fact spontaneously), the major finding here is a warning. We may have strong beliefs that we mathematics teachers have brought about important relational thinking in children but it is often wise to check that this is, in fact, the case. That is, "We taught 'em. Did they learn? What did they learn?"

NOTES

1. O'Brien, T.C. (1986). Memories of George Polya, Mathematics Teaching, 116, pp. 2-6.

2. O'Brien, T.C. (1977). More on Basics. The Genetic Epistemologist, 6 (5).

3. National Test of Basic Skills (1985). American Testronics: Iowa City, Iowa.

4. O'Brien, T.C. and Casey, S.A. (1983a). Children Learning Multiplication: Part I. School Science and Mathematics, 83 (3), pp. 246-251.

5. O'Brien, T.C. and Casey, S.A. (1983b). Children Learning Multiplication: Part II. School Science and Mathematics, 83 (5), pp. 408-412.

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INTRA-INDIVIDUAL DIFFERENCES IN FRACTIONS ARITHMETIC

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Abstract. *Recent evidence indicates that strategy variability in arithmetic is not limited to inter-individual differences and to changes over time. In addition, there are intra-individual differences. Every student has a space of different strategies for each task, and he/she decides anew which strategy to use on each problem. Furthermore, the frequency distributions of strategy spaces are highly skewed, with few frequent strategies and many infrequent ones. The present study presents empirical support for these regularities in the domain of fractions. Current theories of procedural learning fail to explain these regularities. The instructional implication of these findings is that remedial training with respect to particular errors does not necessarily produce correct performance, not even if the training is successful.*

Introduction. During the last three decades it has become increasingly clear that the question "Which cognitive strategy do students use to solve task X?" does not have an answer. First, task analyses have revealed that almost every cognitive task, no matter how simple, *can* be solved with a variety of strategies, and empirical observations have shown that students in fact use different strategies; inter-individual differences with respect to strategy have been found in every task domain in which investigators have looked for them. Second, strategies change over time. Mastery of a cognitive task typically proceeds through successive transformations of incorrect or incomplete strategies into better strategies, and even correct strategies continue to change with further practice. The phenomena of inter-individual differences in strategy and of longitudinal strategy change have been particularly well documented in arithmetic (Goldman, Mertz, & Pellegrino, 1989).

But the evidence from arithmetic also indicates that strategy variability is not limited to inter-individual differences or to changes over time. VanLehn (1982) observed students switch from one (incorrect) strategy to another while solving multi-column subtraction problems, a phenomenon called "bug tinkering" to distinguish it from long-term strategy changes which in VanLehn's terminology are called "bug migrations." Siegler (1987; 1989) and Siegler and Jenkins (1989) recorded both reaction times and protocols from children who solved simple addition and subtraction problems, and found that they have a repertoire of different (correct) strategies for these problems, and that they decide anew which strategy to apply on each trial. Siegler calls this phenomenon "strategy choice." Bug tinkering and strategy choice are similar phenomena: The key point is that there are *intra*-individual differences in strategy, i. e., every student has more than one strategy for a particular task and he/she chooses which strategy to apply on a trial-by-trial basis.

We present empirical evidence for trial-by-trial strategy choice in the domain of fractions arithmetic. In addition, we describe some quantitative

properties of the students' strategy spaces. Finally, we discuss the generality of strategy variability, as well as its theoretical, methodological and instructional implications.

Table 1. A strategy space for the construction of equivalent fractions.

Problem and student's answer[a]	Student's explanation	Strategy
1/5 = 1/10	"Top numbers have to be the same. Bottom numbers doesn't."	$n_2 = n_1$
1/5 = 2/10	"Two times five is ten."	$n_2 * d_1 = d_2$
1/5 = 5/10	"One times five is five."	$n_2 = n_1 * d_1$
1/5 = 5/10	"Because five can go into one, and one can go into five, and five can go into five."	n_2 is a whole number divisor of n_1 and d_1.
12/15 = 2/5	"Because I subtracted ten from this to get five, and I subtracted ten from twelve and I got two."	$d_2 = d_1 - 10$ -> $n_2 = n_1 - 10$
12/15 = 3/5	"Three times five is fifteen."	$n_2 * d_2 = d_1$
12/15 = 32/5	"Five times three is fifteen, and three times twelve is thirty-two."	$x * d_2 = d_1$, -> $n_2 = x * n_1$
4/8 = 8/2	"Bottom number and top number has to be the same."	$n_2 = d_1$
4/8 = 12/2	"I added them [4 and 8] together."	$n_2 = n_1 + d_1$

[a]Problems were presented in the format $n_1/d_1 = ?/d_2$. The student's task was to fill in the missing numerator n_2 in such a way that the resulting fraction n_2/d_2 is equivalent to the given fraction n_1/d_1. The students' numerical answers are underlined.

Method. Seven fifth grade students participated in 13 one-on-one tutoring sessions lasting 20-30 minutes each. The subjects solved problems with a computerized learning environment called the Fractions Tutor (Ohlsson, Nickolas, & Bee, 1987). The Fractions Tutor allows the user to manipulate mathematical expressions as well as concrete representations of fractions. The

user can switch between the different concrete representations with a single mouse click. The Fractions Tutor also provides a *yoking facility* that links a fraction symbol to the corresponding concrete representation. When yoking is activated, changes in the mathematical symbol are automatically reflected in the corresponding concrete representation, and vice versa. If the mathematical symbol is moved about on the screen, the corresponding concrete representation moves in the same way, and vice versa. The philosophy behind the system and the concrete representations have been described in Ohlsson (1987; 1988a).

The students were taught the initial segment of the fractions curriculum: the meaning of the fractions symbol, to compare fractions, the concept of equivalent fractions, to construct equivalent fractions, and to add proper fractions. Their knowledge of these topics was assessed with a paper and pencil, off-line test that was administered both before and after the study. The students were asked to explain the answers they gave to the test problems. For purposes of the present paper, we will focus on three types of problems: (a) *comparison problems*, in which the student was given two fractions and asked which is greater or if they are equivalent, (b) *equivalence problems*, in which the student was given a fraction n_1/d_1 and asked to construct an equivalent fraction n_2/d_2 with a given denominator d_2, and (c) *addition problems*, in which the student was given two fractions and asked to compute their sum. Each student solved twelve comparison problems, four equivalence problems, and four addition problems both on the pretest and on the posttest. The posttest problems were identical to the pretest problems.

Results. The students learned little in the course of the study. The proportion correctly solved problems was .43 on the pretest and .54 on the posttest. A detailed analysis of the reasons for the small improvement has been presented elsewhere (Ohlsson, Bee, & Zeller, 1989). The questions to be answered here concern the students' strategies. Each incorrect answer was classified with respect to strategy on the basis of the student's explanation. The entire strategy space cannot be described in this short paper, but Table 1 shows those strategies that occurred more than once on the equivalence problems, plus a sample explanation for each.

Size and development of the strategy space. Table 2 shows the number of strategies identified for each type of problem. There were 38 different strategies on the pretest, and 36 on the posttest. Hence, the number of incorrect *strategies* did not decrease during the intervention (even though the number of incorrect *answers* decreased a little). This fact does not imply that the strategy space was stable. Only 14 of the strategies which occurred on the pretest also occurred on the posttest, so 22 of the 36 strategies observed on the posttest were either present but invisible during the pretest or else invented in the course of the study.

Trial-by-trial strategy choices. Table 3 shows the proportion of trials on which the student used the same strategy as on the immediately preceeding trial. The overall proportion is .55, indicating that the students switched strategy on approximately half the trials. The students varied widely in their tendency to switch strategies. For example, Student 5 used the previous strategy on roughly two-thirds of the trials; the corresponding figure for Student 2 is roughly one-third. There was no systematic trend towards either more frequent or less frequent strategy shifts from pretest to posttest.

DISCUSSION

The findings here match up with the data gathered by the undergraduates before the present study was undertaken.

Still, this is a preliminary study and it leaves a great deal to be explored.

1. Obviously, the relational justifications (Categories 4 and 5) need further exploration. What, for instance, is the status and evolution of the logical necessity of the justifications? How do they grow out of (if they do) the physical and mental extensions?

2. It is not known whether the low level of relational thinking is the result of a) some developmental limitation, b) an anomalous sample, c) a mathematics curriculum in which relationships received little emphasis, or d) some other reason. If the data gathered from the (comparably aged) British sample significantly exceed those here in the quality of relational thinking, it can be said that there is no developmental limitation on children's relational thinking as assessed by the present tasks.

3. As with previous research by the writer (O'Brien and Casey, 1983 a and b) where 75 per cent or more of American fourth and fifth graders could not construct a multiplicative "story problem" or real-life situation for the stimulus 6 X 3 = 18 (despite the fact that all children could produce that number fact spontaneously), the major finding here is a warning. We may have strong beliefs that we mathematics teachers have brought about important relational thinking in children but it is often wise to check that this is, in fact, the case. That is, "We taught 'em. Did they learn? What did they learn?"

NOTES

1. O'Brien, T.C. (1986). Memories of George Polya, Mathematics Teaching, 116, pp. 2-6.
2. O'Brien, T.C. (1977). More on Basics. The Genetic Epistemologist, 6 (5).
3. National Test of Basic Skills (1985). American Testronics: Iowa City, Iowa.
4. O'Brien, T.C. and Casey, S.A. (1983a). Children Learning Multiplication: Part I. School Science and Mathematics, 83 (3), pp. 246-251.
5. O'Brien, T.C. and Casey, S.A. (1983b). Children Learning Multiplication: Part II. School Science and Mathematics, 83 (5), pp. 408-412.

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INTRA-INDIVIDUAL DIFFERENCES IN FRACTIONS ARITHMETIC

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Abstract. *Recent evidence indicates that strategy variability in arithmetic is not limited to inter-individual differences and to changes over time. In addition, there are intra-individual differences. Every student has a space of different strategies for each task, and he/she decides anew which strategy to use on each problem. Furthermore, the frequency distributions of strategy spaces are highly skewed, with few frequent strategies and many infrequent ones. The present study presents empirical support for these regularities in the domain of fractions. Current theories of procedural learning fail to explain these regularities. The instructional implication of these findings is that remedial training with respect to particular errors does not necessarily produce correct performance, not even if the training is successful.*

Introduction. During the last three decades it has become increasingly clear that the question "Which cognitive strategy do students use to solve task X?" does not have an answer. First, task analyses have revealed that almost every cognitive task, no matter how simple, *can* be solved with a variety of strategies, and empirical observations have shown that students in fact use different strategies; inter-individual differences with respect to strategy have been found in every task domain in which investigators have looked for them. Second, strategies change over time. Mastery of a cognitive task typically proceeds through successive transformations of incorrect or incomplete strategies into better strategies, and even correct strategies continue to change with further practice. The phenomena of inter-individual differences in strategy and of longitudinal strategy change have been particularly well documented in arithmetic (Goldman, Mertz, & Pellegrino, 1989).

But the evidence from arithmetic also indicates that strategy variability is not limited to inter-individual differences or to changes over time. VanLehn (1982) observed students switch from one (incorrect) strategy to another while solving multi-column subtraction problems, a phenomenon called "bug tinkering" to distinguish it from long-term strategy changes which in VanLehn's terminology are called "bug migrations." Siegler (1987; 1989) and Siegler and Jenkins (1989) recorded both reaction times and protocols from children who solved simple addition and subtraction problems, and found that they have a repertoire of different (correct) strategies for these problems, and that they decide anew which strategy to apply on each trial. Siegler calls this phenomenon "strategy choice." Bug tinkering and strategy choice are similar phenomena: The key point is that there are *intra*-individual differences in strategy, i. e., every student has more than one strategy for a particular task and he/she chooses which strategy to apply on a trial-by-trial basis.

We present empirical evidence for trial-by-trial strategy choice in the domain of fractions arithmetic. In addition, we describe some quantitative

Table 2. Number of strategies on pretest and posttest.

Problem type	Pretest	Posttest	Overlap[a]
Comparison	21	22	11
Equivalence	13	11	2
Addition	4	3	1
All problems	38	36	14

[a]Strategies observed both on the pretest and on the posttest.

Frequency distribution of strategies. Figure 1 shows the frequency distributions for the strategies on the pretest and the posttest. The unit of analysis is the individual trial (rather than the student). The distributions are highly skewed. There are many strategies that occur infrequently, but the distributions drop off rapidly towards a thin tail with few but frequent strategies. Although there is little overlap between those strategies that occurred on the pretest and those that occurred on the posttest (see Table 2), the two distributions nevertheless have similar shapes. The two most frequent strategies on the pretest were (a) to compare fractions by comparing their denominators (i. e., the bigger the denominator, the bigger the fraction), and (b) to add fractions by adding both numerators and denominators (i. e., $x/y + u/v = [x+u]/[y+v]$, the so-called *freshman error*). In spite of the large turnover of strategies from pretest to posttest, these two strategies were also the two most frequent strategies on the posttest.

Discussion. In summary, the students in our study were quite creative. They generated a large number of strategies for the different fractions problems; eight per person, on the average. Many strategies were replaced between pretest and posttest. They switched to a different strategy on every other trial, on the average. These data are consistent with the view that each student generates a space of strategies, some correct and some incorrect, for each task, and selects a strategy from that space on each trial, a view of arithmetic performance which Robert Siegler has been advocating for some time (Siegler & Robinson, 1982, pp. 287-299).

Our data contain almost no information about how the strategy choices are made. There is one hint that strategies are chosen on the basis of particular problem features. Consider the fifth strategy from the top in Table 1: *If $d2 = d1 - 10$, then let $n2 = n1 - 10$.* This strategy was only observed on the particular problem $12/15 = ?/5$. It is tempting to hypothesize that this strategy was chosen for this problem *because* the well-learned (and similar) number facts $12-10=2$ and $15-10=5$ come to mind when the problem is perceived.

Table 3.Proportion of all trials on which the strategy used was the same as on the immediately preceeding trial.

Student	Pretest	Posttest	Both tests
1	.60	.47	.53
2	.27	.47	.37
3	.53	.40	.47
4	.73	.67	.70
5	.73	.73	.73
6	.60	.73	.67
7	.40	.33	.37
Average	.55	.54	.55

The frequency distribution for the fraction strategy space is highly skewed, with few frequent strategies and many infrequent ones. Skewed frequency distributions suggest that subjects resolve conflicts between strategies in favor of the most general strategy. This decision criterion imply that general strategies will be choosen often, while highly specific strategies will be chosen only when the general ones do not apply.

Generality. The phenomenon of trial-by-trial strategy variability is not unique to arithmetic. In previous research on verbal reasoning, we observed two main strategies for spatial arrangement problems and succeeded in inducing subjects to shift between strategies within and between trials (Ohlsson, 1984). Similarly, Simon and Reed (1976) observed within-trial strategy shifts on a puzzle solving task. In short, trial-by-trial strategy choices have been observed in at least three rather different types of tasks--arithmetic calculations, verbal rasoning, and puzzle solving--which suggests that the phenomenon is quite general. The phenomenon of a highly skewed frequency distribution has previously been observed by Kurt VanLehn in the domain of subtraction (see VanLehn, 1982, Appendices 2 and 3). VanLehn (1982, p. 44) reports that the three most common subtraction bugs are also the three most stable bugs. Similarly, the two most frequent errors on our pretest were also the two most frequent errors on the posttest. In summary, the phenomena reported here might be general across arithmetic tasks.

Theoretical implication. If trial-by-trial strategy choice is a general phenomenon, a number of novel questions arise for research on arithmetic strategies: How many strategies do students construct for any one task? What are the quantitative characteristics of these strategy spaces? How do strategy spaces change over time? In particular, how do strategy spaces change under the influence of practice, or in response to instruction? Which learning mechanisms can explain the characteristics of strategy spaces? Why do students generate multiple strategies for one and the same task? These questions poses a challenge

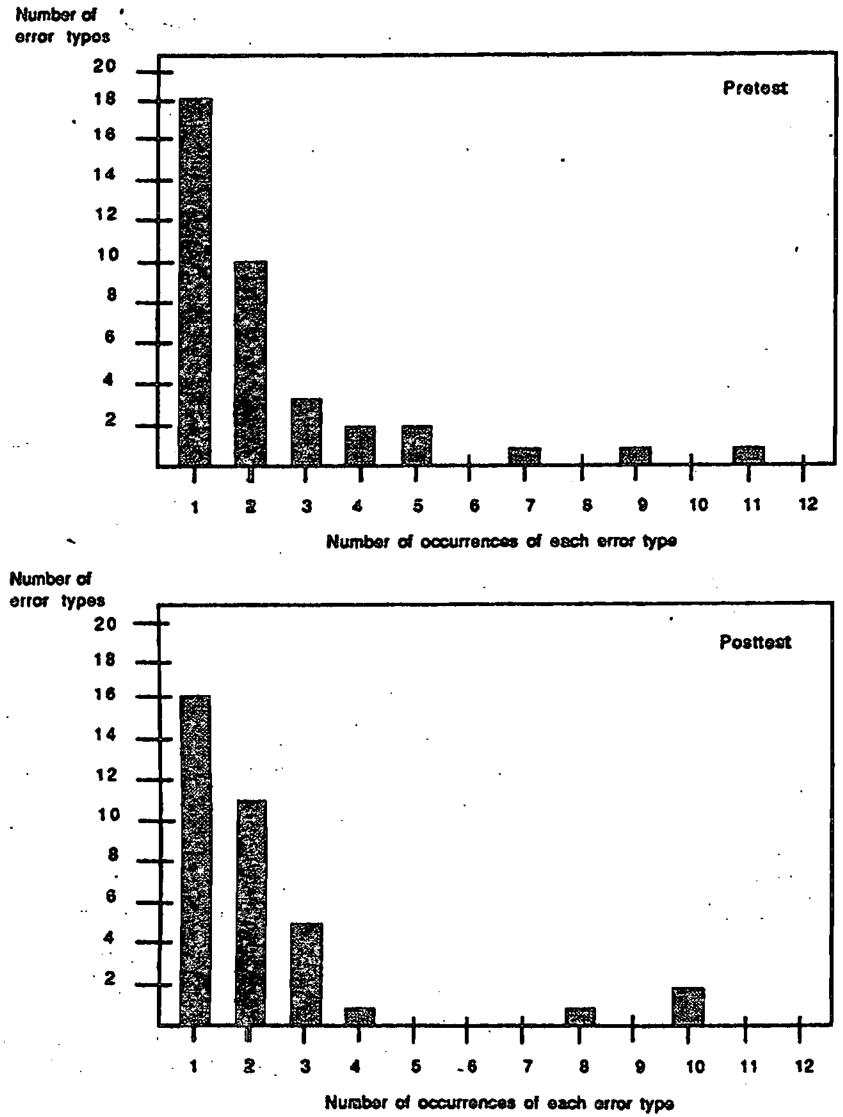

Figure I. Frequency distributions for the strategy space.

for models of arithmetic learning which learn a single strategy for each task (Ohlsson, 1988b; Ohlsson and Rees, in press).

Methodological implication. Trial-by-trial strategy variability implies that quantitative data cannot be aggregated over trials. Enough information has to be collected on each trial to allow identification of which strategy the subject used on that trial. Failure to identify strategies before aggregation can lead to fallacious inferences (Siegler, 1987; 1989).

Instructional implication. The existence of multiple (incorrect) strategies implies that remedial teaching directed at a particular error type should not be expected to lead to correct performance. If the student has, for example, six different strategies for a particular task, four incorrect and two correct, then diagnosing and remedying one of those incorrect strategies leaves the student with more incorrect than correct strategies, *even if the remediation is successful.* Data from experiments with remedial teaching must be interpreted accordingly. How to teach a learner with a rapidly changing space of strategies is an open question.

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References

Goldman, S. R., Metz, D. L, & Pellegrino, J. W. (1989). Individual differences in extended practice functions and solution strategies for basic addition facts. *Journal of Educational Psychology, 81*, 481-496.

Ohlsson, S. (1984). Induced strategy shifts in spatial reasoning. *Acta Psychologica, 57*, 47-67.

Ohlsson, S. (1987). Sense and reference in the design of interactive illustrations for rational numbers. In R. W. Lawler & M. Yazdani (Eds.), *Artificial intelligence and education. Vol. One. Learning environments and tutoring systems* (pp. 307-344). Norwood, NJ: Ablex.

Ohlsson, S. (1988a). Mathematical meaning and applicational meaning in the semantics of quotient concepts. In M. J. Behr & J. Hiebert (Eds.), *Research agenda for mathematics education: Number concepts and operations in the middle grades. Vol. 2* (pp. 53-92). Reston, VA: National Council of Teachers of Mathematics & Hillsdale, NJ: Erlbaum.

Ohlsson, S. (1988b). Computer simulation and its impact on educational research and practice. *International Journal of Educational Research, 12*, 5-34.

Ohlsson, S., Bee, N, & Zeller, P. (1989). *Empirical evaluation of a computer-based learning environment for fractions* (Technical Report No. KUL-89-07). Pittsburgh, PA: University of Pittsburgh.

Ohlsson, S., Nickolas, S., & Bee, N. (1987). *Interactive illustrations for fractions: A progress report* (Technical Report No. KUL-87-03). Pittsburgh, PA: University of Pittsburgh.

Ohlsson, S. & Rees, E. (in press). The function of conceptual understanding in the learning of arithmetic procedures. *Cognition & Instruction.*

, R. (1987). The perils of averaging data over strategies: An example from ildren's addition. *Journal of Experimental Psychology: General, 116,* i0-264.

, R. (1989). Hazards of mental chronometry: An example from children's ıbtraction. *Journal of Educational Psychology, 81*, 497-506.

, R. & Jenkins, E. (1989). *How children discover new strategies*. Hillsdale, J: Erlbaum.

, R. & Robinson, M. (1982). The development of numerical nderstandings. In H. W. Reese & L. P. Lipsitt (Eds.), *Advances in child evelopment and behavior. Vol 16.* (pp. 241-312). New York, NY: Academic ress.

, H. & Reed, S. (1976). Modeling strategy shifts in a problem solving task. *ognitive Psychology, 8*, 86-97.

hn, K. (1982). Bugs are not enough: Empirical studies of bugs, impasses nd repairs in procedural skills. *Journal of Mathematical Behavior. 3*, 3-71.

CONSTRUCTION OF PROCEDURES FOR SOLVING MULTIPLICATIVE PROBLEMS

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Empirical data derived from a research on multiplicative problem solving, with children, between 7 and 12 years, allows me to present the progressive and differentiated manner in which I suppose that the child constructs terms, relations and operations involved in the solution of multiplicative problems.

En investigaciones precedentes Noelting (1980), Gómez Granell (1988), Steffe (1988), Anghilieri (1989), han estudiado resolución de problemas en el "campo conceptual" de la estructura multiplicativa (Vergnaud, 1983), con niños hasta 14 años. Noelting (1980) trabajó problemas de proporcionalidad e identificó estadios de desarrollo. Gómez Granell (1988) trabajó con dos tipos de problemas multiplicativos: de función lineal y de proporcionalidad e identificó procesos aditivos y multiplicativos. Steffe (1988) a partir de episodios de enseñanza, señala la relación existente entre el tipo de unidad que los niños construyen y el concepto de multiplicación que manejan. Anghilieri (1989) trabajó 5 tipos de tareas e identificó estrategias de los niños al resolverlas.

Esta investigación intenta establecer patrones de procedimientos en niños - entre 7 y 12 años - al resolver problemas de tipo multiplicativo; caracterizar su transformación a través del desarrollo, para examinar el paso de los procedimientos aditivos a los multiplicativos; finalmente verificar la relación que se supone existe entre las variaciones en los contextos experimentales que las tareas conforman y las variaciones en los procedi-

mientos de los niños al resolverlas.

Trece tareas relativas a tres tipos de problemas multiplicativos especifican las variaciones de los contextos experimentales: seis tareas de función lineal que varían en el contenido - compraventa y mezcla - y en el rango numérico que manejan (3,7,8); cuatro tareas de proporcionalidad, en el contenido - compraventa y mezcla - y en el carácter de los operadores escalar y funcional (Vergnaud, 1983): exacto (2,4) (2,6) e inexacto (2,3) (2,5); tres tareas de producto cartesiano con el mismo contenido - paredes - que varían en el rango numérico (3x4, 3x8, 7x8). Las tareas se presentaron en entrevistas individuales en las cuales se utilizaron enunciados sencillos y materiales que el niño manipulaba.

El trabajo de investigación, que se encuentra en proceso, ha posibilitado construir un modelo de análisis que permite establecer la manera cómo los niños resuelven los diferentes tipos de tareas, y diferenciar en el análisis los datos que manejan, del modo cómo obtienen el resultado (Orozco, Valencia, Bedoya, 1990). El modelo surge del análisis de las respuestas de 10 niños (2 por nivel de edad) al resolver las trece tareas. Así mismo, el modelo ha permitido crear un programa para computador que actualmente se utiliza para analizar las producciones de una muestra de 100 niños - 50 en Cali, Colombia y 50 en San José, Costa Rica - al resolver las mismas tareas.

Este artículo analiza, a través de estudios de caso, la manera como se supone el niño construye los términos, las relaciones y

operaciones que le permiten resolver tareas multiplicativas de función lineal.

CONSTRUCCION DE TERMINOS, RELACIONES Y OPERACIONES AL RESOLVER TAREAS MULTIPLICATIVAS

Para resolver multiplicaciones, el sujeto debe manejar simultáneamente dos términos, llamados factores: el mutiplicando y el multiplicador, los cuales se componen entre sí mediante una operación binaria. Verganud (1983) señala que, para resolver problemas de regla de tres (para esta investigación tareas de función lineal), y aún, para la misma operación binaria de multiplicación, el sujeto pone en relación cuatro términos - tres datos: 1, a, b y la incógnita X - en dos espacios de medida. Igualmente menciona que en su resolución, además de la operación binaria, el sujeto puede presentar otros dos tipos de operaciones unarias: una escalar y otra funcional.

Al analizar producciones de niños ante tareas que utilizan material, es igualmente necesario tener en cuenta la relación de los objetos con los términos del problema. En el caso que analizo[1], el "1" es un confite que el entrevistador entrega al niño; "a", el precio del confite, configurado por una colección de monedas que el niño entrega al entrevistador; "b" corresponde a una colección de confites que el entrevistador entrega al niño; y "X", a la pregunta: "cuánto tendrías que pagar por estos

[1] Se analizarán las producciones de niños al resolver una tarea de función lineal. La tarea es: Te doy este dinero para que compres confites (Muestra un confite al niño). Un confite vale $3 (Niño y entrevistador intercambian monedas y confite). Cuánto tendrías que pagar por estos confites? (Se le entregan 7 confites en la mano).

confites?". El anterior análisis permite suponer que estas tareas exigen al niño manejar cuatro términos, los objetos relacionados con ellos y las relaciones y operaciones entre los términos.

Anghilieri (1989) describe minuciosamente cómo construyen los niños el multiplicando y en relación con el multiplicador, señala que los niños usan el segundo factor para "terminar el patrón numérico que construye, en el punto requerido" (p. 382) Quisiera mostrar que aunque inicialmente, este número, y no factor, desempeña esta función, igualmente inicia, en la mente del niño, la construcción del segundo factor, o sea, del multiplicador. Así mismo, quisiera describir cómo construyen los niños las relaciones entre los términos de la tarea y las operaciones que realizan al resolverlas. Examinemos las respuestas de dos niños, de diferentes edades a la tarea de función lineal, previamente descrita.

José (1o. de primaria, 7 años) (Desliza sobre la mesa, con el dedo índice de la mano derecha, uno a uno, tres confites, separándolos de la colección que el entrevistador le ha entregado) Dice: 3, 6, 9; (desliza otro), diciendo: 10, 11, 12; (desliza otro) y dice: 13, 14, 15; (desliza otro) y dice: 16, 17, 18; (desliza otro) y dice: 19, 20, 21. E: Cuántos confites hay? (tapando con su mano el montón de 7 confites). José: No sé.

Claudia (2o. de primaria, 8 años) (Separa dos confites de la colección que recibe; separa otro confite y golpea la mesa con tres dedos; separa otro y repite el mismo movimiento. Repita la

misma actividad hasta coger el séptimo confite). Dice: 21. E: Cómo supiste que son 21? Claudia: Conté los dedos de 3 en 3 y me dió 21. E: Cuántos confites hay? (tapando los confites sobre la mesa). Claudia: 7.

José explicita su manera de resolver la tarea. Su procedimiento puede ser: asigna 3 a cada uno de los tres primeros confites e itera el operador "+3"; a partir del cuarto confite, asigna 3 unos e itera el operador "+3 veces 1", completando la secuencia hasta 21. En el espacio de medida de los confites, José no maneja operador escalar alguno, simplemente desliza cada confite hasta que termina su traslado. En el espacio de medida del precio, inicialmente itera el operador escalar "+3" y posteriormente el escalar "+3 veces 1". El traslado del último confite, le indica que debe terminar su iteración, pero no registra el número de confites que traslada.

En tanto Claudia, de manera implícita, monitorea el número de confites en correspondencia con sus patrones de dedos. Examinemos su procedimiento. Ella resuelve la tarea mentalmente, a medida que manipula los objetos que intercambia con el entrevistador. Sin embargo, las actividades verbales y motoras que realiza, el tiempo de solución y el resultado que obtiene, permiten inferir su procedimiento: asigna 6 a los dos primeros confites[3]; cuenta uno a uno los confites a medida que los separa; e itera "tres', número que mantiene en sus dedos y dice haber "contado". Parece ser que al contar uno a uno los confites, e iterar el tres, Claudia establece la relación funcional (1,3)

[3]Por falta de espacio no se analiza esta asignación.

(2,6) (3,9)... (7,21). Claudia no maneja el operador funcional "x3". Si lo manejara, no necesitaría iterar la relación (1,3). Tampoco maneja el operador escalar "x7"; en su defecto, maneja los operadores escalares "+1", en el espacio de medida de los confites, y "+3", en el del precio. Para obtener el resultado, parece ser que la niña opera simultáneamente la relación (1,3) y los operadores escalares "+1", y "+3", y construye la relación funcional (1,3) (2,6) (3,9)... (7,21). Es esto último, lo que le permite determinar y registrar el número de confites que manipula y manejar los tres datos, resolviendo la pregunta.

El siete, no sólo constituye el límite de las veces que Claudia debe contabilizar el 3, en el patrón de dedos, sino que además corresponde al número de confites. Durante el proceso que sigue para llegar al 21, el "siete" funciona implícitamente. Ella sólo lo expresa cuando el entrevistador le pregunta por el número de confites. Posiblemente, durante el proceso el siete no es expresado, porque la niña está centrada en el 3 que itera. Sin embargo, su respuesta al entrevistador permite suponer que, simultáneamente con la iteración, hace seguimiento (Steffe, 1990) al número de confites contados. De esta manera, comienza a funcionar, en la mente de Claudia, el segundo factor, que permite la multiplicación.

Si se compara el análisis de las producciones de Claudia y José se encuentran niveles progresivos en el manejo de la multiplicación. En el espacio de medida del precio, los dos manejan el operador escalar "+3", Claudia durante todo el proceso y José, al inicio del mismo: posteriormente, José maneja el escalar "+3 veces 1" En el espacio de medida de los confites, solamente Claudia maneja el operador escalar "+1". Como José no lo maneja, entonces no puede determinar el número de confites que traslada. José y Claudia manejan la relación (1,3), Claudia como parte de una relación funcional. José llega al resultado manejando la relación (1,3) e iterando los escalares "+3" y "+3 veces 1", en el mismo espacio de medida. Claudia llega al resultado manejando simultáneamente la relación (1,3), los operadores escalares "+1" y "+3" en dos espacios de medida diferentes y construyendo la relación funcional (1,3) (2,6) (3,9)... (7,21), que es la que permite a Claudia decir, al final de la entrevista, el número de confites que ha manejado y que el entrevistador tiene tapados. Las operaciones que estos niños utilizan son unarias. Apenas empiezan a construir el segundo factor que posibilita la operación binaria. Solamente cuando maneja en su mente simultáneamente, los dos factores, el niño utiliza procedimientos multiplicativos y con ellos, manejar la operación binaria.[3]

BIBLIOGRAFIA

Anghiliere, J. (1989). An investigation of young children's understanding of multiplication. *Educational Studies in Mathematics*, 20, 367-385.

Gómez-Granell, C. (1988). Representación y simbolización en el marco de problemas multiplicativos. Universidad de Barcelona. Tesis doctoral. No publicada.

Noelting, G. (1980). The development of proportional reasoning and the ratio concept: Part II. *Educational Studies in Mathematics*, 11, 331-363.

Orozco, M., Valencia, M. E. & Bedoya, E. (1990). Patrones de solución de problemas multiplicativos en niños de 7 a 12 años.

[3] La falta de espacio impide analizar procedimientos propiamente multiplicativos.

Educación Matemática, Vol. 2, 1., 51-51.

Steffe, L. P. (1988). Children's construction of number sequences and multiplying schemes. In J. Hiebert & M. Behr (Eds.), *Number Concepts and Operations in the Middle Grades*. Reston: National Council of Teachers of Mathematics (pp. 119-140).

Steffe, L. P. (1990). Children's construction of meaning for arithmetical words a curriculum problem. In D. Tirosh (Ed.), *Implict and Explict Knowledge in Mathematics and Science*. Tel Aviv: Tel Aviv University (in press).

Steffe, L. P. & von Glasersfeld, E. (1985). Helping children to conceive of number. *Recherches en Didactique des Mathématiques*, 6, 269-303.

Vergnaud, G. (1983). Multiplicative structures. In R. Lesh & M. Landau (Eds.), *Acquisition of Mathematics Concepts and Process*. New York: Academic Press (pp. 127-174).

TRANSFER IN LEARNING 3D REFERENCE SYSTEM : FROM INTERACTION WITH COMPUTER SOFTWARE TO COMMUNICATION OF SPATIAL CHARACTERISTICS

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Can knowledge constructed during the interaction with a computer software be reinvested and adapted to solve new problems ? In this research, a learning sequence is designed; it has the 3D reference system as object. The first part is based on the production and transformation of plane representations of 3D configurations, by using computer (CAD software) as a tool. The second part doesn't involve computer use; it aims at a transfer, adaptation and evolution of the knowledge acquired. Experimentation has been conducted in a computer workshop of a French school. A clinical analysis of data showed that transfer processes occured, where the concepts previously constructed have been adapted to the new situation. This analysis allowed also to detect the mechanisms of such a transfer, and the aspects of the situations that affected them.

One of the major concerns when computer is used as a tool in math teaching is the problem of transfer : Can knowledge constructed during the interaction with a computer software be reinvested and adapted to solve new problems ? Arguments have been given by opponents in the debates concerning the effectiveness of computer use : "knowledge constructed in a computer context is likely to remain related to this context". Very few are the studies that investigated the mechanisms of adaptation that occur when concepts developed in a computer context are to be used in a different context. For this purpose, we designed a learning sequence having the 3D reference system as object. The first part is based on producing and transforming plane representations of 3D configurations, by using computer as a tool (CAD software). The second part doesn't involve computer use; it aims at a transfer, adaptation and evolution of the knowledge acquired.

Why 3D reference system ?

By observing the school-books and the teaching practice, we can notice that the *reference system* notion is introduced, since the intermediate level, as an established fact. It isn't constructed as a solution to specific problems necessitating organization and structuration of physical space. As this notion is introduced, pupils are suddenly projected in *analytic geometry*. A new language, a new system of symbolic representations are used, fixed by the teacher, but not constructed by the pupils. Algebraic relations are defined and used to replace geometric relations between the elements of a spatial configuration. Then, the geometric activity is transformed into a calculatory activity, in which the interpretation at the geometric level is neglected. With this modification of the nature of "geometric" activity, the *meaning link* between spatial and algebraic frameworks is at risk being suddenly broken.

In *3D geometry*, such a problem is even more accurate : there is a strict separation between activities of concrete manipulation and activities using abstractions and theorizations. One of the main reasons of such an aggravation is the difficult access to spatial situations. Our research assumes that computer can help to overcome some of the problems related to representation of 3D objects : the perception and interpretation of graphic representations. The first part of the learning sequence uses this type of access to spatial situations. The second part uses concrete models made of cardboard.

Context of the research

The research exposed in this paper adopts, as theoretical framework, the constructivist theory of learning. Our method is to construct several "situations" linked to each other, aiming at a learning process, through transfer and adaptation of previous knowledge, in order to solve each new problem within the new constraints. Some of these problem-situations use computer graphic facilities. We adopt the hypothesis that there exists a close interaction between the acquisition by the pupils of the concerned geometric knowledge and their activity of discovering the functioning mode of used software.

Learning situations have been constructed and experimented, in the context of a computer workshop in a french school, with pupils of the fourth intermediate class (14-16 years). Pupils work by pairs, each pair having a computer machine available (a Macintosh). 24 students (12 pairs) were involved. During the experimentation, we recorded the steps of pupils' work (as computer files and observation notes); we recorded also their dialogues; we used this data for a clinical analysis of the evolution of their strategies.

Particularly, this paper aims at presenting concisely some results of this clinical analysis. We will be concerned with the transfer between two representation systems, corresponding to two of the problem-situations. The following are the tasks to be performed by the students in each one of them :

i- "Mac Space" task : Using the software Mac Space, construct the graphic representations of spatial configurations (polycubes, surfaces in scale, etc...).

ii- "Models" task : Write a message, the shortest possible, to describe a given model, in such a way that your classmates can reconstruct it, by only following the instructions given in your message. Drawings are forbidden. Students are provided with : a model made of cardboard, a pen, a ruler, a set-square, and a "Description" sheet.

Note that the description phase of the "Models" situation is not the last one in the sequence. It is followed by two other tasks : The reconstruction of models described by others, and the suggestion of a new description for the model decoded. In this paper, we will only consider the first task, hoping to expose a complete analysis of the three phases in future papers.

Learning process along the previous situations*

The current state of students' knowledge is a result of the whole learning process that took place during the previous situations, not only the "Mac Space" situation. The following is a quick review of the evolution of students' abilities along the previous problem-situations :

- Evolution of the perceptive ability, through tasks where the students had to construct and to interpret graphic representations of spatial configurations. Perceptive ambiguities were purposely introduced in some of the tasks, in such a way that perception plays alternatively a double role : as an obstacle or as a means of control.
- Evolution of the ability to coordinate the views in space, and progressive awareness of the third dimension in each one. i.e. depth for the face view, level for the top view,....
- Evolution towards an analytical conception of space : from spaces where the geometric elementary objects are cubes, through spaces where the elementary objects are planes and straight lines, to spaces where the elementary objects are points.
- Organization of the space according to different structures (topological, by parallel planes and euclidean) and construction of several reference systems, related to these structures. The last reference system is constructed through the detection of the reference system controlling the space of a computer software : "Mac Space".

In order to understand the nature of the reference system detected and constructed in the previous situation, we present below some aspects of its conceptual analysis :

"Mac Space" : reference system and construction procedures

Mac Space is a conversational graphic editor that works on Macintosh. It helps user to construct representations of 3D objects by constructing views (top, face and side views). The representation in perspective appears progressively in a fourth window, called 3D.

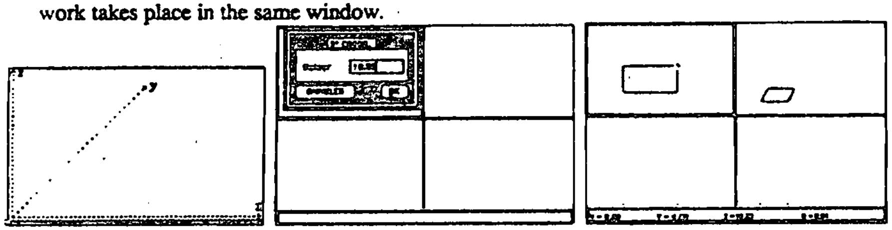
The space of the software is controlled by an implicit orthonormal reference system, composed by three non-materialized axes, two by two perpendicular: Ox, Oy, Oz. The first figure below simulates the three virtual axes of this system in the 3D window.

Since the selection of a graphic tool, the coordinates of the current point represented by the cursor are displayed dynamically (they change together with the current position of the cursor). The space is considered, in each one of the windows, as an addition of a privileged direction, perpendicular to a privileged plane. It is isomorph in each one to a non-associative product of three unidimensional spaces: (Ox.Oy).Oz in the window of the top view, Ox.(Oy.Oz) in that of the side view and (Oz.Ox).Oy in that of the face view.

* For more information about the sequence of problem-situations, see Osta 1988. For more information about the process of learning "Mac Space" reference system, see Osta 1989.

The following is the procedure of construction, with Mac Space, of a rectangle parallel to one of the planes of the 3D reference trihedron. For such a task, student has to :

- identify and select the appropriate window in which the facet has to be constructed;
- determine the coordinate of the plane containing the rectangular facet with respect to the privileged plane of the window and communicate it to the computer through the command "3º coord." (2d fig. below); since then, the 3º coordinate will keep this value as long as the work takes place in the same window.



- move the cursor then validate (by clicking) the position of a vertex of the rectangle to be constructed; while the cursor is moving, its coordinates (x,y,z) are dynamically displayed;
- move the cursor again, to reach the position of the vertex, opposite to the one already validated. While the cursor is moving, values of the relative coordinates of the new point with respect to the validated one are dynamically displayed. The changing coordinates displayed are then the dimensions of the current rectangle (3d fig. above).

Analysis of the "Models" task :

Objectives of the situation

Formulation : The sequence of activities with computer aimed at confronting the students with situations where they learn, by doing, concepts related to 3D reference system. Language was almost not used by the students, since they interacted with the computer through selection of commands and graphing. The current problem-situation aims at confronting the students with the use and the formulation of the concepts, in such a way that these concepts are detached of the context where they have been constructed. It aims at making the students able to use these concepts explicitely in appropriate situations, and to communicate them to others in an appropriate way. Note that formulation is not merely a simple association of codes and symbols to the elaborated concepts; there is an interaction between formulation and learning process.

Transfer and adaptation : Along the learning sequence with the computer, the problem-situations aimed at a progressive structuration of space, and the construction of several reference systems. Particularly, during the Mac Space task, the students built knowledge about the reference system of this software. Our hypothesis is that this knowledge can be reinvested and adapted to new situations where this software is not involved.

Characteristics of the "Models" situation can be classified in two categories :

1) Elements inciting transfer : The task has similarities with the previous one : Student has to communicate spatial characteristics to a constructor (computer or classmate). So, in both situations, he has to translate his knowledge of spatial relations into instructions for the reconstruction. Moreover, the composition of the objects to be described has similarities with those represented graphically in the previous situations : the fundamental geometric object in these models is the rectangular surface. Almost every rectangular component is parallel to one of the three basic directions in space (horizontal, normal or frontal). But the models have also differences with the objects involved in the previous situations; contrary to the first thought, these differences also incite transfer. Indeed, one of them is that each model contains one oblique rectangular component, that is not parallel to any of the three basic planes; an other difference is that the numeric relations between the components of the models are much more diverse than in the regular objects previously involved. We made these choices to make more difficult any description with natural language, based on properties of parallelism, perpendicularity or regularities. For such components, more than one vertex must be determined. We suppose that such constraints will incite students to abandon natural language and to use more symbolic and numerical ones, the most probable being the reference system previously learned.

2) Elements necessitating adaptation :

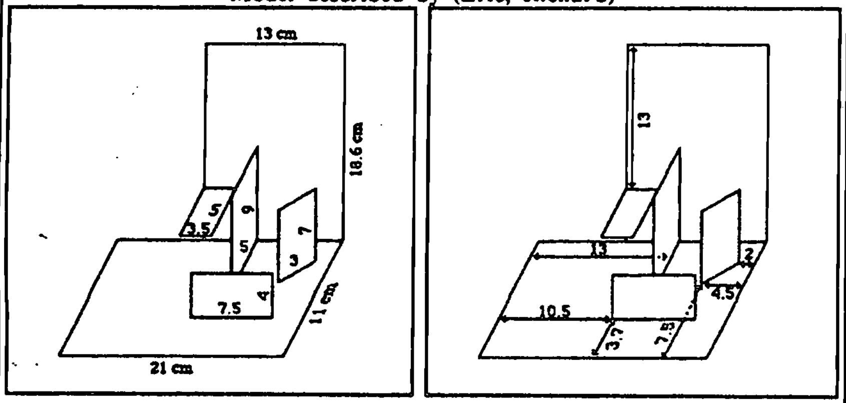
i) From computer microworld to physical microspace : Objects involved in the previous situations are symbolic representations of spatial configurations. At the beginning of a graphic representation task, the viewpoint is determined. Moreover, in Mac Space, the reference system is subjacent and controls the functioning mode of the software. The previous tasks aimed, in fact, at a detection and a mental construction of this already existing reference system. In the "Models" situation, the students have physical objects of the "microspace", that provides freedom of manipulation and viewpoint modifications. The situation doesn't involve any predetermined reference system. Students have to make the decision of using a reference system, then to determine its components and characteristics.

ii) From graphic representations to language : The "Models" task involves a new symbolic code to represent the objects. Students can not any more benefit of the figurative analogy of graphic representations (drawings are forbidden). The only symbolic code used now is language (natural, mathematical,...). We suppose that the language used will evolve towards a symbolico-mathematical language, all along the analysis of the object to be described and the trials to express its complex relations.

iii) From interaction to communication : Compared to the tasks with computer, the communication situation creates an important constraint : During the work with computer, a permanent interaction exists between the student and the machine; The student has an instantaneous feed-back after any operation : error messages, requests for information,

graphics, etc... This provides him with an important means of evaluation of his procedures. In the "Models" situation, such interactivity is absent. The description of the model is only communicated to the constructors at the end, without any possible feed-back. The validation of the system adopted is then postponed until the whole description is completed. This will incite students to search for unique and coherent systems. It will incite also a progressive evaluation by the students of their own systems, by anticipating the interpretation that the constructor can make of their message.

Model described by (Eric, Richard)



Description : *[[[Base of the figure]]] [[[Level 0 , rectangle [[[(11 ; 21)]]] L. 21 , D 11]]] [[[Level 0 . D 11 , H]]] [[[Top view : rectangle (21 , 11) ; level 0 rect]]]*

1 - rectangle for the base (21 ; 11)

2 - rectangle ⊥ to R. of B. placed at the depth 11 and at 8 from the left corner . coordinate (18.6 ; 13)

3 - rectangle ⊥ to R. of B. placed at the de. 3.7 and at 3 from the right edge and at 10.5 from the left edge. C(7.5 ; 4)

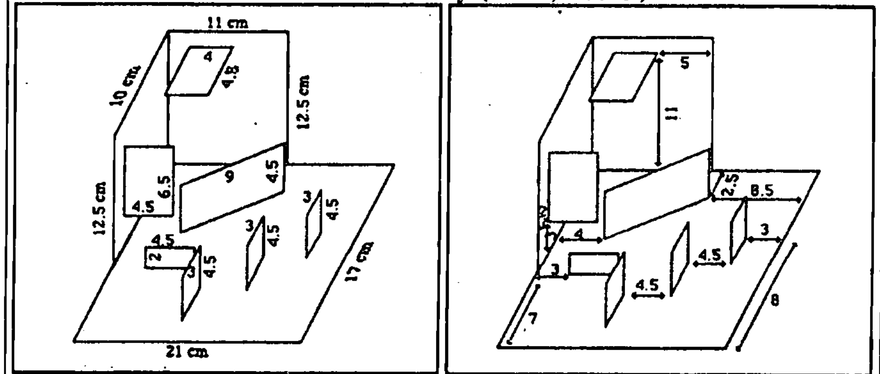
4 - rectangle ⊥ to R. of B. placed at the de. 7 (left) and 9 (right) and at 1.5 from the right edge and at 17 from the left edge.

5 - rectangle ⊥ to R. of B. placed at the de. 6 [[[and 17]]] and at 8 from the right edge and at 13 from the left edge.

6 - rectangle // to R. of B. , of height 5, width 3 and length 5.

N.B. : The messages written by the students (in italics) have been translated from French. We tried to be most accurate in conveying exactly the sentences used by the students, their trials and the scratched parts. The text inserted between *"[[["* and *"]]]"* has been scratched by the students.

Model described by (David, Rachel)



Description : *[[[Base of 21 and 17 cm]]]*

Consider a reference system in space O i , j , k (abscissa , ordinate , depth).

The different values correspond to two opposite vertices of a rectangle to do.

A = 0 ; 0 ; 0 and B = 21 ; 0 ; 17 => 1st rectangle

C = 17.7 ; 0 ; 7.8 D = 17.7 ; 3.8 ; 10.6 => 2nd rectangle

E = 13.5 ; 0 ; 5 F = 13.5 ; 3.70 ; 8 => 3rd /G = 9.3 ; 0 ; 2 H = 9.3 ; 10 ; 5 => 4th

I = 3 ; 0 ; 7.3 J = 7.5 ; 1.3 ; 7.3 => 5th /K = 3.6 ; 0 ; 10.8 L = 11.3 ; 3.4 ; 14.5 => 6th

M = 0 ; 4 ; 3.8 N = 4 ; 9.3 ; 8 => 7th /O = 2 ; 10.3 ; 10.8 P = 5.5 ; 10.3 ; 17 => 8th

Q = 0 ; 0 ; 7 R = 0 ; 12 ; 17 => 9th /S = 0 ; 0 ; 17 T = 11 ; 12 ; 17 => 10th

Some results

In the current situation, reference systems acquire a new meaning : as a means for communication of spatial information. In the students' messages, we have detected indices indicating a transfer of knowledge acquired during the previous situations, especially those using the software Mac Space. Even though they are different, the reference systems used by the students contained such indices. For commodity reasons, let's consider examples corresponding to two cases that we consider representative of two different orientations of evolution in the students' processes.

Eric and Richard used the same process used in Mac Space to situate the position in space of a rectangular component : determine the "level" of its plane, determine the position of one vertex, then give the dimensions of the rectangle; these dimensions are called "coordinates", which recall the dimensions expressed by Mac Space as the relative coordinates. They have even used the same language used by Mac Space, or developed by the class community on the basis of its functioning mode : level, depth, 3rd coordinate, etc... In their trials, we can notice more relationships with Mac Space style than in their final description. By analysing the evolution of their processes through their dialogues, we found two reasons for this fact :

i- These two students had many problems in their discovery and acquisition of the Mac Space reference system itself. In the previous situation, they couldn't achieve the task. Irrespective of the direction of the rectangle to be constructed, they proceeded on the top view. In the present message, they tried to situate the rectangles with respect to the base.

ii- The model they had to describe is relatively simple : it contains only 6 components. This made weak the constraint against natural language. When the students began to face the difficulties mentioned above, they changed their procedure that used Mac Space style.

David and Rachel constructed a 3D reference system, by designating an origin and 3 axes. By its orientation, this reference system reminds that of Mac Space. The analysis showed that the students built their procedure on the knowledge acquired with Mac Space, together with that acquired in the math class about 2D reference system. This explains the use of the words "abscissa", "ordinate". The students called the 3º dimension "depth", the same word used during the Mac Space sequence, for the "3º coord." in the face view. The students moved progressively towards an analytic procedure, by determining the positions of two opposite vertices of each component, base included. Note that this definition of a rectangle by two opposite vertices is the one used by Mac Space.

Conclusion

The clinical analysis allowed to detect the phenomena of transfer and adaptation of concepts, from computer context to situations not involving computer. We saw that such phenomena involve many aspects of the cognitive activity : a) The types of reference systems constructed (3D usual reference system, or juxtaposition of 2D reference system with a 3º dimension, or in the worst cases natural language description). b) The vocabulary used (depth, level, 3º coordinate, views, ... c) The procedures used to define the position and the dimensions of each component . d) The mental processes, such as the process of coordination between the different dimensions for example. e) The task of reconstruction by someone else of the model was often identified by the students to a task of constructing its graphic representation according to the rules of Mac Space. At some places, their messages sound like instructions given to someone, so that he can represent the model graphically with the software.

Finally, we found a close interaction between the description systems adopted and the composition of the object described.

Bibliographic references :

Osta I., 1988, L'ordinateur comme outil d'aide a l'enseignement. These de l'universite Joseph Fourier, Grenoble.

Osta I., 1989, Graphic constructions with computer to learn 3D reference system. Proceedings of PME 13 (Paris), volume 3, pp. 65-73.

LEVELS OF KNOWLEDGE ABOUT SIGNED NUMBERS: EFFECTS OF AGE AND ABILITY

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This research suggests a theoretical analysis of children's knowledge about signed numbers together with the operations of addition and subtraction. This analysis has resulted in a description of knowledge levels in the dimension of a number-line representation and in the dimension of a quantity representation. This taxonomy is aimed to aid in the understanding of the complexity of this domain and the children's route towards gaining more knowledge. Interviews with children of different abilities are used to explore the power of the analysis.

INTRODUCTION

During their elementary school years children face a very demanding task: They are required to keep accommodating their number system to include more sets of numbers. We have described their difficulties and the duration of this process for the assimilation of decimals (Nesher and Peled (1986), Resnick et al (1989)), where we have also observed the influence of the child's previous knowledge on the perception of new information. This research will deal with the acquisition of signed numbers' knowledge, and will try to outline possible levels of knowledge on the child's route to understanding addition and subtraction of these new numbers.

Previous research on signed numbers (Peled et al (1989)), in which we have interviewed children at a large age range (1st, 3rd, 5th, 7th and 9th grades), and research done by other researchers (Davis et al (1979), Murray (1985), Human and Murray

(1987)), are the basis on which this research has built in trying to understand what children know, and how their state of knowledge determines their ability to expand their signed numbers' knowledge. This paper will, thus include two parts:

1. A theoretical part which describes levels of signed numbers' knowledge.

2. An empirical part which includes children's interviews.

The second part will give us some indication as to the power of the theoretical description, as the latter will be used in the design and the analysis of the interviews.

LEVELS OF KNOWLEDGE

Children's descriptions of how they perceive negative numbers and what operations with signed numbers mean to them, has lead to a description of their knowledge on two dimensions: a quantitative dimension and a number-line dimension. As will be seen, a child can hold more than one image of signed numbers together with the operations, and might use different images with distinct types of number problems.

THE NUMBER LINE DIMENSION

Level 1: The child knows that there also exist numbers, called negative numbers, to the left of zero on the number line. These numbers are almost a reflection of the "regular" numbers, as they start with "minus one" and continue left by having the word "minus" added to the "regular" number: minus two, minus three, etc. The definition of the order relation is extended to apply

also to the new numbers: Given two numbers , any of which can be regular or negative, the one which is further to the right is the bigger number.

Level 2: The child is ready to do a simple extension of the operations of addition and subtraction. In performing subtraction this means that the child agrees to go further left even beyond zero, on the number line, when a larger number is subtracted from a smaller number. In performing addition on the number line, it means that the child is ready to step to the right even when the starting point is a negative number.

Level 3: The child is ready to further extend the definitions of addition and subtraction which involves stepping along the number line. It is accepted that

addition and subtraction involve opposite directions, but another factor appears: the signs of the numbers which are added or subtracted. There are two worlds: a positive world on the right and a negative world on the left. Just as addition meant going towards the larger numbers in the positive world, it also means going towards the larger-in-negativity numbers in the negative world, i.e. one has to move towards the left when adding within this world. A similar argument results in moving right when one performs subtraction in the negative world.

One should note that the operations have been extended to pairs of signed numbers having the same sign, and are not yet defined for numbers of different signs.

Level 4: By now the child does not have to regard the similarity in signs and the type of signs both numbers have, but can simply refer to the second number to determine whether the movement on

the number line will be to the right or to the left. When the second number is positive, one faces the positive direction, thus going right for addition and left for subtraction. When it is negative, one faces the negative world, going towards it to the left for addition and away from it to the right for subtraction.

THE QUANTITY DIMENSION

Level 1: Just as there are regular numbers, representing amounts of things, there exist negative numbers, which are also amounts of things. These are, though, things of an unfavorable characteristic, such as amount of owed money. Because of this negative connotation, the order relation on these numbers is defined in an inverted way to the regular numbers, i.e. the larger the amount, the smaller is the number, as it stands for a worse-off state.

Level 2: The child is able to extend the operation of subtraction so that it allows for a larger (natural) number to be subtracted from a smaller one. This is done by taking away the available amount, and figuring out the amount missing to complete the operation. The latter is actually the absolute value of the result, which gets labeled by a minus sign to represent this state of debt or deficiency.

A note: The order between level 1 and 2 in this dimension might be reversed.

Level 3: The child is willing to extend the definitions of addition and subtraction, of adding amounts and taking away amounts, to apply to negative amounts as well. This means that a negative quantity can be taken away from a negative quantity

as long as there is enough to take away from. A negative quantity can also be added to a negative quantity, resulting in the increase of the negative amount. However, amounts of different types cannot be handled at this level.

Level 4: The quantities do not have to be of the same type. The effect of the operation is determined by the operation and the sign of the second number only, i.e. one has to consider whether taking away or bringing more of the relevant quantity results in a better (meaning mapping into addition of a natural number) or worse (meaning mapping into subtraction of a natural number).

THE EMPIRICAL STUDY

The purpose of this study was to utilize the former analysis in looking at the range of knowledge levels of children who have already learned to perform addition and subtraction of signed numbers.

Twenty children, who had learned this topic the year before, were chosen for the study. The children were chosen from two sixth grade classes so that there were ten good students and ten weak students. Using a structured interview including number problems, which were built based on the levels' analysis, each student had one interview session lasting about twenty minutes.

These interviews are now being analyzed, still some preliminary results can already be reported. The levels of these children's knowledge are surprisingly low, even for the ten good students. Most of the weak students do not even remember that they have studied this topic, but also some of the good students, who claim

they remember studying operations with signed numbers, do not admit it for all types of number combinations (e.g. a student might claim that she has not learned problems such as (+2)-(-4), while admitting to have studied all other types of problems).

This poor performance has its good sides. Since students do not remember how to handle the different types of number problems, they have to reconstruct their knowledge. In Brown and VanLehn's terms (1982) they are making repairs of incompletely remembered procedures, in many cases building them from scratch. This gives us the chance to follow the mechanism of repair, which sometimes lets us into the process of progression from one level to another.

One of the good students, for example, uses a number line definition of subtraction and shows how he moves on the number line to get correct answers to the problems: 3-7 and (-4)+2, which involve a Level 2 extension. He then proceeds to cope with some operations with signed numbers: 6+(-4) and (-4)-(-2). In both cases he claims that since one gets to a negative number by going left on the number line, then adding such a number should similarly mean going left. This helps him relate to the (-4) in both problems. There still remains the issue of taking away (-2) from the -4 and this is solved by claiming that now after walking 4 steps left, one has to walk left 2 steps less, resulting in going in the positive direction. However, consecutive problems: (+4)-(-2) and 2-(-3) are incorrectly solved, leaving him at Level 3, where numbers can be handled as long as they are from the same world (and there is enough to step away from).

Among the weak students some strange repairs occur. One student, who can draw a correct number line, but cannot show regular subtraction on the number line, is asked to solve 3-7 . He claims that this is impossible because "It is impossible to do something [from a number] small[er] than what you take away. Nothing will come out of it". When asked in a similar problem, (-4)+2, to think again, he suggests borrowing from the 2 and getting 1.6 (2.0-[0.]4=1.6).

Another student, who cannot even order the numbers correctly, performs a somewhat similar repair. He solves 3-7 by changing the 3 into 30, saying "Now I borrow (moves a ten to the ones column). Ten minus seven is three (writes 3 in the ones column) and here it's two (writes 2 in the tens column)". He gets 3-7=23 and though, when asked about it, he doesn't seem satisfied with this answer, he can only suggest more-of-the-same-type repairs, such as changing the numbers into 0.3-0.7.

Some of the weak students do manage to make good repairs. Thus, for example, a student who has no number line representation, but does have a quantity representation of negative amounts, can add correctly: (-3)+(-5)=-8.

To summarize, when the data is analyzed, it will probably place most of the students within the first two levels of the number line representation, and first three levels of the quantity dimension. A few of the good students might fall at the third level of the number line dimension This would mean that a further testing of the levels with older children is needed. The existing data, however, seems to indicate that the levels' analysis is useful in giving us a tool to understand the

complexity of this knowledge domain and to enable us to diagnose what a specific child knows and where that might lead him in terms of his ability to progress to higher levels of knowledge.

REFERENCES

Brown, J. S. & VanLehn, R. R. (1982) Towards a generative theory of "bugs". In T. P. Carpenter, J. M. Moser & T. A. Romberg (Eds.) Addition and Subtraction: A Cognitive Perspective. pp. 117-135 Hillsdale, NJ: Erlbaum.

Davis, R. B., McKnight, C. Parker, P. & Elrick, D. (1979) Analysis of student answers to signed number arithmetic problems. The Journal of Children's Mathematical Behavior. V 2 pp. 114-130.

Human, P. & Murray, H. (1987) Non-concrete approach to integer arithmetic. Proceedings of the 11th International Conference of PME. V 2 pp. 437-443.

Murray, J. C. (1985) Children's informal conceptions of integer arithmetic. Proceedings of the 9th International Conference of PME. pp. 147-153.

Nesher, P. & Peled, I. (1986) Shifts in reasoning (the case of extending number concepts). Educational Studies in Mathematics. V 17 pp. 67-79.

Peled, I., Mukhopadhyay, S. & Resnick, L. B. (1989) Formal and informal sources of mental models for negative numbers. Proceedings of the 13th International Conference of PME. V 3 pp.106-110.

Resnick, L. B., Nesher, P., Leonard, F., Magone, M., Omanson, S. & Peled, I. (1989) Conceptual bases of arithmetic errors: the case of decimal fractions. Journal for Research in Mathematics Education. V 20 N 1 pp. 8-27.

Représentations du problème de mathématiques chez des enfants de 7 à 10 ans

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Abstract

The research reported here is a part of a more important one, some aspects of which have been presented in PME 14. In this part, we wonder if we can see, among young children, the birth of some difficulties that we observed with older ones. We chose to do interviews with about 8 years old pupils. We describe here results on their conceptions about mathematics problem. An example : pupils think that a text without questions is a problem because "there are numbers" or "it's us that have to answer". We analyze these results with the concept of "didactic contract".
An english version of this text will be available in June at Assisi.

1. Problématique.

Le travail présenté ici fait partie d'une recherche plus importante qui porte sur l'enseignement des mathématiques à des enfants en difficulté à l'école, principalement issus de milieux sociaux défavorisés. De nombreux travaux ont étudié la relation entre origine sociale des élèves et réussite scolaire mais il en existe peu qui prennent en compte le contenu à enseigner. Notre projet était de cerner concrètement comment certains élèves passaient dans le système scolaire à côté de certains apprentissages de base et de chercher des variables intermédiaires possibles pour expliquer les différences d'apprentissage à partir de situations en apparence identiques. Pour cela, il nous fallait entrer au cœur de la relation didactique qui fait intervenir maître, élèves et contenu pour analyser comment les interactions entre maître et élèves qui se nouent (ou non) dans les classes, à propos d'un savoir précis, peuvent engendrer l'échec de certains élèves.

Nous avons exposé un des aspects de cette recherche, centré sur le rôle du maître, à PME 14 [5]. Rappelons ici les principaux axes d'interprétation, côté élèves, des difficultés observées :
- projet général qu'ils forment vis-à-vis de l'école,
- rapport qu'ils entretiennent avec les mathématiques,
- interprétation des attentes de l'enseignant,
- investissement dans les situations et reconnaissance de leur enjeu,
- création de représentations mentales au cours des situations d'action.

Une étude de cas (observation pendant 6 mois d'un élève de 4ème année de primaire) nous a permis de comprendre l'irrégularité des performances souvent observée chez ces élèves qui n'arrivent pas toujours à utiliser des connaissances qu'ils possèdent et qui perdent facilement le fil de ce qu'ils sont en train de faire. On peut parler de connaissances à la fois "floues et rigides" :
- floues parce qu'ils sont rarement sûrs de ce qu'ils avancent,
- rigides parce qu'ils ont beaucoup de mal à changer de point de vue, de stratégie et à saisir les indications qu'on leur donne : les informations qui ne rentrent pas exactement dans la procédure envisagée par l'élève le déstabilisent et il se met à dire n'importe quoi.

L'enfant que nous avons observé, comme beaucoup d'élèves en difficulté, recherche les algorithmes, plus sécurisants, moins fatigants et, surtout, qui le libèrent de la responsabilité de ce qu'il avance. Tout se passe comme s'il redoutait de prendre la responsabilité de ses connaissances et

se reposait sans cesse sur le discours du maître qu'il perçoit comme une suite de recommandations et de règles sur la conduite à tenir. Son rôle à lui est d'écouter et de faire ce que dit le maître.

Ces caractéristiques sont assez fréquentes chez les élèves en difficulté à l'école mais, au cours de nos observations, nous avons aussi rencontré un autre type d'élève en difficulté : celui qui ne peut accepter des règles ou des algorithmes qu'il ne comprend pas complètement et qui reste sur des procédures archaïques de traitement. Les élèves qui sont dans ce cas ont au contraire tendance à éviter les algorithmes mais ils risquent aussi de se dire rapidement qu'ils ne comprennent rien en mathématiques parce qu'ils ne comprennent pas tout, puis qu'il ne faut pas chercher à comprendre et finalement rejoindre le comportement précédent.

Nous avions travaillé jusque là avec des élèves de fin d'école primaire et nous avons cherché à voir si, chez des élèves plus jeunes, les attitudes face aux mathématiques et aux problèmes que nous avions observées, étaient déjà présentes ou si on pouvait trouver en germe ce qui les ferait se manifester plus tard, et c'est à cette partie de la recherche qu'est consacré le présent article.

Nous allons maintenant exposer la méthodologie adoptée. Nous donnerons ensuite quelques-uns des résultats obtenus, avant de les interpréter en termes de contrat didactique.

2. Méthodologie.

Nous avons élaboré un questionnaire qui a servi de base à des entretiens individuels avec des élèves de 2ème année de primaire (âge normal de 7 à 8 ans).

Elaboration du questionnaire

Notre objectif était de recueillir des déclarations des élèves concernant leur rapport à l'école et aux mathématiques, sur un plan assez général, et de les mettre en relation avec leur attitude et leurs performances sur des contenus mathématiques. Nous avons donc prévu des questions générales et d'autres sur le contenu.

Etant donné l'âge des élèves, une première question visait à vérifier que l'élève savait reconnaître une leçon de mathématiques. Dans les autres questions générales, nous interrogions l'élève sur son intérêt pour les mathématiques, sur ses pratiques en classe ou à la maison. Nous voulions aussi déceler les idées qu'il pouvait avoir sur l'utilisation des mathématiques hors de l'école et sur l'apprentissage des mathématiques, et l'interprétation qu'il faisait des paroles des adultes à ce sujet. Voici, à titre d'exemple, la question qui s'est révélée la plus significative pour la mise en relation avec le contenu : "Que penses-tu qu'il est le plus important de faire pour être bon en mathématiques? Et à ton avis, qu'en pense la maîtresse ?"

Les contenus que nous avons retenus sont d'une part la maîtrise des nombres et notamment la numération qui nous paraît un point fondamental sur lequel s'appuient les apprentissages numériques et même, pour une bonne part, algébriques ultérieurs, d'autre part la résolution de problèmes qui est une caractéristique de l'activité mathématique.

La numération est un des enjeux essentiels de ce niveau, sur lesquels la maîtresse nous a dit beaucoup travailler, notamment par des jeux d'échanges. Nous avons choisi de faire compter les élèves

mentalement, pour voir s'ils utilisaient les dizaines pour compter plus vite.. Nous avons également posé à certains élèves un problème dont la résolution reposait directement sur la numération.

Sur le problème, nous avions deux objectifs :

- d'une part, cerner ce qu'est pour l'élève un problème de mathématiques,
- d'autre part observer la manière dont il aborde la résolution d'un problème.

Nous pensions déceler l'idée que les élèves se faisaient du problème de mathématiques en leur demandant d'en fabriquer un eux-mêmes et en analysant la forme et le contenu des questions posées. Pour avoir une certaine unité et pour éviter que les élèves ne restent "secs", nous avons choisi de leur demander de fabriquer un problème à partir d'un contexte fourni. Le choix des questions des élèves nous informerait aussi sur la structuration qu'ils faisaient des données de l'énoncé.

Nous avions prévu le texte suivant : "Dans une classe, on organise un goûter. On a décidé d'acheter des croissants et des tablettes de chocolat. Les croissants sont vendus par sachets de 10. Les tablettes de chocolat sont vendues par boîtes de 5. Dans cette classe, il y a 25 élèves. Invente des problèmes à partir de cette histoire et essaie d'y répondre."

Le premier entretien nous a montré que, contrairement à notre attente, pour l'élève interrogée, le texte fourni était déjà un énoncé de problème. Nous avons donc ensuite commencé par demander à chaque élève si l'histoire qu'on lui lisait était ou non un problème de mathématiques. (Ils ont été unanimes, la réponse a été oui pour tout le monde !) Nous leur demandions ensuite si on leur avait posé des questions, éventuellement lesquelles, puis d'en poser eux-mêmes et d'essayer d'y répondre. S'ils ne pouvaient en poser, nous leur en proposions une.

Choix de la classe et du mode de questionnement

Pour une étude exploratoire comme la nôtre, les entretiens individuels se révélaient préférables parce qu'ils apportaient des informations nettement plus précises. Sur la partie mathématique, on peut par exemple juger de la vitesse de calcul, poser des questions complémentaires pour comprendre les causes d'erreur des élèves, donner une petite aide pour voir si l'élève peut l'utiliser. Sur les questions périmathématiques, on peut poser la question de façon très ouverte et avoir des réponses spontanées qui traduisent peut-être plus exactement la pensée de l'enfant, avant de faire des suggestions si l'élève n'a pas d'idée. Cependant, dans les entretiens, les informations ne sont peut-être pas toujours aussi fiables qu'on le souhaiterait : sur les questions générales notamment, le problème se pose de trier dans les réponses des élèves entre ce qu'ils pensent et ce qu'ils pensent qu'on attend d'eux, ce qui pourrait être "une bonne réponse". Il est en effet connu[1] que les jeunes enfants ont tendance à classer les adultes à qui ils ont affaire dans le cadre scolaire, notamment dans une situation de questionnement expérimental, en deux catégories : celui qui fait apprendre dans le cas où le savoir en jeu paraît nouveau, celui qui évalue si le savoir en jeu paraît ancien. Dans leurs réponses, il nous faudra donc faire la part de ce qui peut être l'envie de se conformer à ce qu'ils perçoivent de notre attente, prolongeant ainsi le contrat didactique habituel.

Notre propos était de faire une étude clinique et non une étude comparative : nous avons choisi d'interroger les 21 élèves d'une classe d'une école de recrutement populaire, où avaient eu lieu une partie de nos observations précédentes. Nous avons également demandé à la maîtresse un

enregistrement d'une séquence en classe sur la numération, et nous disposons aussi d'un entretien avec elle. Nous pouvons donc mettre en relation ce que disent les enfants avec ce que dit l'enseignant.

Les entretiens ont eu lieu en mai 1988[2]. Les mêmes élèves ont été interrogés en français. Une comparaison des résultats était prévue mais n'a pas encore eu lieu

Méthode de dépouillement et d'analyse

Nous avons procédé à un dépouillement par question. Cela nous permet de dégager des tendances au niveau de la classe qu'on peut rapprocher des déclarations du maître. Nous avons ensuite situé chaque élève sur les axes identifiés pour chaque question. Même si notre cadre théorique, qui nous a permis d'élaborer le questionnaire, nous donne des axes pour le dépouillement, les catégories précises ne peuvent se déterminer qu'a posteriori, au vu de ce que nous recueillons réellement. Pour le problème, par exemple, nous avons finalement dégagé les axes suivants :

1 - pourquoi est-ce un problème ? (donc ce qu'est supposé être un problème de mathématiques)
2 - l'élève peut-il poser une question ? 3 - quelles questions sont posées ?
4 - quelles questions implicites les élèves se posent-ils et hypothèses implicites font-ils sur la situation
5 - qu'est-ce qui relève de la logique du quotidien dans les questions, les remarques ou les démarches
6 - résolution du ou des problèmes posés. 7 - difficultés éventuelles sur les contenus.

3. Résultats concernant le problème.

1. Pourquoi est-ce un problème de mathématiques ?

Pour certains élèves, le problème semble se reconnaître à l'existence d'une question, soit parce qu'ils croient en avoir décelé "il y a des questions", soit parce qu'ils cherchent à répondre à une question qu'ils se sont formulée implicitement "oui, parce qu'il y en a que 15 et il faut encore 10 croissants sinon on partage les tablettes de chocolat". Certains expriment même l'idée d'une question avec les éléments nécessaires pour la réponse "on donne les prix" dit un élève qui ajoute ensuite "c'est pas un problème parce qu'on donne pas l'argent qu'il fallait".

Pour d'autres, c'est plutôt l'idée d'une histoire contenant des données numériques qui permet de reconnaître un problème. Cela peut être très explicite "les calculs où on met des phrases, c'est des problèmes", ou beaucoup plus vague "on dit que les croissants sont vendus par sachets de 10 et les tablettes de chocolat par boîtes de 5".

Pour d'autres encore, le problème est plutôt lié à l'idée qu'on peut calculer, qu'ils le disent ou qu'ils indiquent une opération "oui, parce qu'ils ont 25 et y'a 10 croissants... les tablettes de chocolat sont vendues par boîtes de 5 et puis ça fera 15", "on fait une opération entre 10, 5 et 25".

Il se peut aussi que l'élève reconnaisse le problème au fait qu'il est de sa responsabilité de faire quelque chose "(c'est un problème parce que) c'est nous qui devons répondre".

Ces critères ne sont pas exclusifs et certaines déclarations des enfants peuvent être rattachées à plusieurs. De plus, d'autres élèves y adhèreraient sans doute si on les leur proposait.

Certains élèves ne peuvent pas donner d'argument, ils se réfèrent à un problème qu'ils ont traité en classe qui sert de modèle, mais par le contexte.

Pour un élève, c'est même le sens courant du mot problème qui est retenu "quand il a un problème, il demande au maître et il lui explique et sinon, il demande à son camarade."

La majorité des élèves reconnaît, quand on le leur demande, qu'il n'y a pas de question posée ici. Mais certains pensent qu'on leur a posé une question, d'autres ne savent pas. L'élève qui prend problème au sens de difficulté pense qu'il n'y a pas de question dans un problème.

Quelques élèves émettent cependant des doutes, se sentent mal à l'aise ou trouvent ce problème un peu différent de ce qu'on leur pose d'habitude, deux dès le début "peut-être, oui", d'autres quand ils ont réalisé qu'il n'y avait pas de question ("c'est pas vraiment un problème" "pourquoi ?" "parce qu'il y a pas une question", un autre "on doit dire combien vont-ils dépenser?")

Pour certains élèves, un problème qui parle d'acheter des choses doit, de façon plus ou moins explicite, poser des questions sur le prix. Un élève trouve même une réponse.

2. Possibilité de poser une question et nature des questions posées

Huit élèves ne peuvent pas poser de question, même si certains d'entre eux répondent à des questions implicites : certains élèves ne peuvent pas séparer la question de sa réponse. On peut d'ailleurs penser que certains élèves ne savent pas très bien ce qu'est une question, quelques-uns par exemple proposent des réponses quand on leur demande de poser une question. Neuf élèves posent des questions, même si c'est parfois après avoir déclaré ne pas pouvoir en poser.

Les questions posées ne sont pas toutes de même nature et pas toutes conformes à ce qu'on peut attendre classiquement dans un problème de mathématiques :

- deux élèves posent des questions pertinentes pour l'organisation du goûter mais non pour un problème de mathématiques,
- une élève pose une question acceptable pour un problème de mathématiques mais inutile ici (réponse donnée dans le texte), tout en sachant que c'est une question inutile
- trois élèves posent une question acceptable pour un problème de mathématiques mais dont les données ne permettent pas de trouver la réponse, cela concerne le prix. Deux de ces élèves savent cependant que les données ne permettent pas de répondre, le troisième pense pouvoir y répondre.
- quatre élèves posent une question pertinente pour un problème de mathématiques construit sur ce contexte, même si cette question contient parfois une hypothèse implicite comme celle-ci : "combien il faut en acheter des autres ? des autres tablettes de chocolat et croissants ?"

3. Questions implicites. Hypothèses implicites ou interprétation inexacte des données.

Nous appelons question implicite une question à laquelle l'élève fournit une réponse sans l'avoir formulée. Ces questions sont le plus souvent liées à une hypothèse implicite, faite par l'élève, c'est-à-dire une interprétation ou un ajout de données qui lui permet de donner du sens à la question à laquelle il répond. Par exemple, beaucoup d'élèves supposent qu'on a acheté un paquet de croissants et une boîte de chocolats et se demandent s'il y aura assez pour toute la classe ou ce qu'il faudra acheter d'autre, en faisant souvent l'hypothèse supplémentaire qu'il faut donner une chose à chaque enfant : "10 et 5, ça fait 15, donc il n'y aura pas assez", "il faut acheter encore 5 tablettes de chocolat et 5 croissants". D'autres, qui veulent interpréter le problème en termes de prix, comprennent qu'un paquet de croissants coûte 10F. Certains sont bloqués dans la résolution parce qu'ils pensent à la fois qu'il faut que tout le monde ait la même chose et qu'il ne faut pas de reste.

4. Logique du quotidien

Des propriétés du contexte qui apparaissent à leurs yeux très importantes font qu'il existe des questions et des réponses que certains enfants ne peuvent pas envisager. Par exemple, six élèves sont très préoccupés par l'idée d'avoir juste assez avec un partage égal et voudraient "acheter 20 croissants au supermarché et 5 à la boulangerie", ou "demander au monsieur d'ouvrir un sachet et de donner 5 croissants" ou "acheter 2 sachets de croissants et des petits gâteaux" pour compléter.
Quand on demande à un élève comment sont vendus les croissants, il répond "en payant". Un autre à qui on demande ce qu'on pourrait faire, répond "chercher de la boisson".

Certains élèves sont ainsi plus préoccupés de traiter le problème matériel du goûter que le problème mathématique auquel il est prétexte.

5. Résolution des problèmes et difficultés particulières

Onze élèves résolvent au moins un problème seuls ou avec une aide légère, même si deux d'entre eux ont d'abord eu besoin de beaucoup d'aide pour en résoudre un autre.

Les nombres en jeu sont très petits. La principale difficulté rencontrée est la confusion objet - paquet pour 6 élèves. L'un d'eux semble aussi avoir des difficultés de calcul avec les nombres de cette taille. Ils ont tous des niveaux faibles en numération.

4. Analyse en termes de contrat didactique

L'analyse des résultats obtenus pour chacune de ces rubriques au niveau de la classe nous permet de dégager des pistes sur ce qu'est un problème de mathématiques pour les élèves de cette classe, sur les phénomènes imbriqués qui interviennent dans la résolution de problème pour les enfants de cet âge. En d'autres termes, elle nous renseigne sur la naissance du contrat didactique autour du problème de mathématiques.

Revenons d'abord sur la notion de contrat didactique. Cette notion, introduite par G. Brousseau, trouve en partie son origine dans les notions de contrat social de J. J. Rousseau et de contrat pédagogique de J. Filloux. G. Brousseau [2] le définit comme ce qui est spécifique du contenu, de la connaissance mathématique visée dans "la relation qui détermine - explicitement pour une petite part, mais surtout implicitement - ce que chaque partenaire, l'enseignant et l'enseigné a la responsabilité de gérer et dont il sera d'une manière ou d'une autre, responsable devant l'autre." Dans le contrat didactique est ménagée une place pour l'élève et une place pour le maître : chacun a une part de responsabilité dans la progression des connaissances. Comme le remarque Chevallard [3], de l'entrée dans le contrat didactique procède un savoir qui ne peut être mis en texte puisque les clauses du contrat ne sont jamais énoncées, qu'elles sont "partout tacitement admises et reconnues" (terminologie empruntée à Rousseau) et aussi qu'elles sont universellement violées ! En effet, le contrat didactique n'est pas une réalité statique mais en continuelle évolution : en se modifiant, il fait évoluer les significations des contenus et des formes de l'échange didactique. Le maître est amené à violer le contrat didactique, à provoquer des ruptures, vers le haut pour faire avancer la connaisssance, ou vers le bas pour faire des rappels par exemple. C'est pour lui un moyen de gérer le "temps didactique". Le contrat didactique évolue donc en même temps que les connaissances des élèves. Pour que l'élève apprenne, il est nécessaire qu'il entre dans le contrat didactique et en suive toutes les évolutions.

Le contrat didactique en vigueur dans une classe à un moment donné dépend de la connaissance en jeu mais il dépend aussi des représentations[3] des élèves et des enseignants sur cette connaissance et sur la manière de se l'approprier. Les maîtres ne peuvent pas mettre en place et gérer un contrat didactique qui soit en désaccord avec leurs convictions profondes. Les élèves ne peuvent sans doute pas non plus entrer aussi facilement dans n'importe quel type de contrat didactique. Le milieu social d'origine peut avoir une influence sur la facilité d'entrée des élèves dans tel ou tel contrat didactique et dans l'adhésion à son évolution.

Avant d'interpréter les résultats en termes de contrat didactique, il nous faut préciser de quel contrat nous parlons, celui qui est en vigueur dans la classe ou celui qui se noue avec les expérimentateurs. Nous avons été présentées comme des dames qui viennent les interroger pour savoir ce que les enfants pensent des mathématiques. Des doutes subsistent pour eux sur nos attentes: sommes-nous là pour leur apprendre quelque chose ou pour les évaluer ? est-ce qu'on est vraiment en mathématiques ? quel est le problème qu'on leur demande de résoudre, l'organisation d'un goûter ou un calcul ? Sur quoi les attend-on ? D'une part l'élève a tendance a prolonger avec nous le contrat didactique habituel, d'autre part, il n'est pas sûr de sa validité. C'est dans les effets de cette ambiguïté et dans les ruptures que nous provoquions par rapport au contrat habituel que nous pouvons repérer quelques "clauses" de ce contrat habituel. Nous décrivons ci-après les ruptures majeures identifiées :

- Nous provoquions une première rupture du contrat didactique en demandant à l'élève de fabriquer lui-même un problème alors que son rôle se borne en général à résoudre. De plus, pour produire lui-même un problème, il ne pouvait pas forcément imiter ce qu'il connaissait, par exemple un problème déjà traité en classe, puisque le contexte et les données lui étaient imposées.
- Il y avait rupture à un deuxième niveau puisque les problèmes pertinents qu'on pouvait poser à partir du texte étaient des problèmes de division et donc que la réponse au problème ne pouvait pas être un résultat d'opération, du moins à leur niveau.
- Il apparait aussi une rupture due à l'ambiguïté entre le contrat didactique et le contrat expérimental[4]. Le contexte de l'histoire est tiré de la vie quotidienne, aucune question n'est posée et le texte leur est proposé dans une situation en marge du scolaire, alors que le lien éventuel avec la classe n'est pas clair pour eux ; les élèves se trouvent donc devant un champ de possibilités extrêmement large pour poser des questions, d'autant plus large qu'il leur est difficile de situer les attentes de l'interrogateur. La frontière entre les mathématiques et le reste redevient floue.

Il s'est avéré à travers ces ruptures que le contrat didactique à propos du problème de mathématiques était à la fois plus fort et plus vague que nous ne le pensions. D'une part, les élèves de cette classe ne savent pas très bien à quel registre faire appel, d'autre part, ils essaient de jouer au mieux leur rôle d'élève. Ce qui ressort au niveau de la classe, c'est qu'un problème de mathématiques, c'est quelque chose qui engage la responsabilité de l'élève : il doit fournir une réponse. De plus il y a des phrases et des nombres, il faut donc chercher soi-même dans le texte les données à combiner et les opérations à faire, contrairement à ce qui se passe pour d'autres exercices de mathématiques. Mais ce n'est pas à lui de poser des questions, même si cela fait partie de son rôle à d'autres moments de son travail scolaire, par exemple pour montrer qu'il sait ce qu'est une question. Ce n'est pas à lui non plus de savoir si ce qu'il propose est juste ou non, pour cela il attend le verdict du maître ou de l'expérimentateur.

Cependant, il semble que la plupart des élèves manquent de repères pour sélectionner les données pertinentes et les mettre en relation ou même pour en prendre réellement connaissance, et que la tâche qu'ils se donnent face à un problème est plutôt de fournir une réponse à l'aide des nombres du texte sans même se soucier de la question posée. Il n'est pas étonnant dans ces conditions qu'il se trouve des enfants de cet âge pour donner "l'âge du capitaine"[5], sans même qu'on le leur ait demandé, et trouver le prix du goûter quand les données portent sur tout autre chose !

Conclusion

Une des raisons des difficultés que nous avions repérées avec des élèves plus âgés paraissait être l'impossibilité à se représenter le problème et le manque d'anticipation. Il semble que, pour la plupart des élèves de cette classe, il y ait bien anticipation de questions. Le problème est que, souvent, les questions prévues ne sont pas celles qui intéressent le maître, ou qu'elles sont mêlées à d'autres questions qui ne relèvent pas des mathématiques.

Les résultats sur la numération sont relativement liés à ceux sur la résolution de problème, mais pas à la capacité de poser des questions (il est vrai que très peu d'élèves l'ont fait). De plus, la réussite en numération et en résolution de problèmes sont toutes les deux liées au fait de mettre en avant, pour devenir bon en mathématiques, le travail de l'élève comme "apprendre ses leçons".

Par ailleurs, sur l'ensemble des élèves, on a une image de ce que dit l'enseignant qui coïncide assez bien avec ce qu'il dit lui-même mais certains enfants n'intègrent que les recommandations de surface comme "être sage, bien écouter", alors que d'autres en retirent des choses plus fondamentales au niveau de leur propre responsabilité, voire de l'attitude de recherche. La cohérence avec ce qui se passe dans la famille a sans doute un rôle dans cette différence, mais d'autres facteurs interviennent aussi, par exemple l'attitude du maître qui est peut-être différente suivant les élèves, notamment quand ils sont considérés comme des bons élèves ou comme des élèves en difficulté. Nous en avons d'ailleurs quelques exemples dans la séquence enregistrée.

Références
[1] Abric J. C. (1987) *Coopération, compétition et représentations sociales* Del Val (Suisse).
[2] Brousseau Guy (1987) Fondements et méthodes de la didactique des mathématiques *Recherches en didactiques des mathématiques n°72.* La Pensée sauvage, Grenoble.
[3] Chevallard Yves (1983) *Remarques sur la notion de contrat didactique* IREM de Marseille.
[4] Equipe "Elémentaire" de l'IREM de Grenoble. (1980). Quel est l'âge du capitaine ? *Bulletin de l'A.P.M.E.P. n°323* APMEP, Paris
[5] Perrin-Glorian M.J. Reflexions sur le rôle du maître dans les situations didactiques à partir du cas des élèves en difficulté *PME 14, Mexico* p.209-216. (english version available).
[6] Robert A. et Robinet J. (1989) Représentations des enseignants de mathématiques sur les mathématiques et leur enseignement *Cahier DIDIREM n°1* IREM Université Paris 7.
[7] Schubauer-Leoni M.L. (1986). Le contrat didactique dans l'élaboration d'écritures symboliques par les élèves de 8-9 ans. *Interactions didactiques n°7* Genève

[1] voir par exemple les travaux de M.L. Schubauer-Leoni [7].
[2] Je remercie vivement J. Robinet qui m'a aidée à interroger les élèves
[3] Nous prenons ce terme au sens défini par A. Robert et J. Robinet [6] à partir de la notion de représentation sociale (voir par exemple Abric [1])
[4] voir M.L. Schubauer-Léoni [7]
[5] voir APMEP [4] et Chevallard [3]

Teaching as meta-communication

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Abstract

In this paper, I outline a distinctive aspect of teacher discourse, namely that of making meta-communicative remarks, and identify it with a central aspect of the teaching enterprise. Illustrations are given from mathematics classes (and further ones will be offered in my talk) and potential consequences explored.

Metaphors are one way of exploring aspects of the complex task that is teaching. When looking at the discourse generated about teaching, both academically and popularly, a number of such metaphors can be seen at work, each with its own imagery and rhetoric. Examples include teaching as gardening (kindergarten, growth, nurturing, but what about 'weeds'?), as doctoring (remediation, diagnosis, prescription, but misconception as illness?), as coaching (basic skills, exercises, repetition, competition and performance) and as acting (classroom roles, stage managing events, teacher as director or actor, pupils as actors or 'audience').

Metaphors act to highlight possibly unnoticed aspects as well as suppressing others, stressing and ignoring according to the perception offered by seeing X *as* Y. In this paper, I want to explore another metaphor, one offered by educational linguist Michael Stubbs (1983, my emphasis): "There is a sense in which, in our culture, teaching *is* talking". Reading this as a metaphor affords a perception on teaching, and encourages the question: Is it any kind of talking which is teaching, or are there particular kinds?

This moves us into the arena of discourse analysis, which since the mid-1970s has been used to explore aspects of classroom discourse, among others, and to highlight certain normative aspects of language use. One early 'finding' by Sinclair and Coulthard (1975) was the almost incessant repetition of the sequence I(nitiation) – R(esponse) - F(eedback) in teacher-pupil exchanges. (A more detailed account of this sequence and mathematics teachers finding ways to escape from it is given in Pimm, 1987.) Discourse analysis provides some content-free classifications, but in mathematics education I am interested in the interaction of content with these customary styles of communicating that classrooms seem to engender. The option of attending to the form of an utterance over its

...t as a means of directing or shifting attention is the particular aspect I wish to focus on here.

Further examples using discourse analysis *within*mathematics education can be found in Ainley's (1988) work on the varied functions of questions and Edwards and Mercer's (1988) examination of the rhetoric of 'progressive' education in English primary schools, in particular, the latters' focusing on a disparity between the level of freedom accorded at the level of action and that at the level of discourse and generation of knowledge. In particular, they draw attention to the various indirect teacher devices for constructing the 'common knowledge' in the classroom - controlling the flow of conversation; determining who is allowed to speak, when and about what; use of silence to mark non-acceptance of a pupil's conversation; reconstructing recaps of what has been said, done or 'discovered'.

One sense of unease that discourse analysis has engendered is its necessary ignoring of the content of the discourse, of what is being siad or taught, in favour of the form. That is a consequence of its enterprise. One claim I wish to make in this paper is that this ignoring of content and focusing on form is one hallmark of a teacher. In *Up the Down Staircase*, novelist Bel Kaufman (1964) tells poignantly of a male English teacher receiving a love letter from a female pupil and the way he copes with this is by correcting the spelling and grammar, making some remarks about clichéd use of language, and then returning it to her with a grade. Such an extreme instance of attending to form over content merely serves to highlight one important possibility for unusual discourse in classrooms: precisely because the actions would have been entirely unexceptional and appropriate within a classroom setting.

In an early paper, entitled 'Organizing classroom talk', Stubbs (1974) offers the notion that one of the characterising aspects of teaching discourse as a speech event is that it is constantly organised by meta-comments, namely that the utterances made by pupils are seen as appropriate items for comment themselves and, in addition, that many of the meta-remarks are evaluative. He comments:

> The phenomenon that I have discussed here under the label of meta-communication, has also been pointed out by Garfinkel and Sacks (1970). They talk of "formulating" a conversation as a feature of that conversation.
>
> > "A member may treat some part of the conversation as an occasion to describe that conversation, to explain it, or characterise it, or explicate, or translate, or summarise, or furnish the gist of it, or take note of its accordance with rules, or remark on its departure from rules. That is to say, a member may use some part of the conversation as an occasion to *formulate* the conversation."
>
> I have given examples of these different kinds of "formulating" in teacher-talk. However, Garfinkel and Sacks go on to point out that to explicitly describe what one is about in a conversation, during that conversation, is generally regarded as boring, incongruous, inappropriate, pedantic, devious, etc. But in teacher-talk, "formulating" is appropriate; features of speech do provide occasions for stories worth the telling. I have shown that teachers do regard as matters for competent remarks such matters as: the fact that somebody is speaking, the fact that another can hear, and whether another can understand. (p. 23)

In Stubbs' paper itself, the examples he uses to exemplify his various types of speech acts undertaken by teachers are all from a lesson involving an English language teacher working with a group of teenage pupils who are non-native speakers. Nonetheless, despite the fact that in such a class matters of language and the form of utterances being obvious foci for attention, I am taken with his ostensible characterisation (in part) of teaching in terms of making meta-comments. In a group discussion of teaching strategies, Elizabeth Richardson (1967) offered one of coming into the classroom to find an offensive drawing on the board. In addition to the strategies of ignoring the content of the message and rubbing it off, and the confrontational approach of demanding "Who wrote that?", is the possibility of commenting on the utterance, by drawing attention to the fact that someone has written on the board and asking "I wonder why they did that". In this way, the message on the board becomes an object of study and speculation for the class rather than a direct message to the teacher by an unknown communicant. Richardson writes: ... try to get the class to talk about the feelings or preoccupations or anxieties that give rise to such a demonstration of hostility against an adult." (p. 100)

Below I give two short extracts from secondary mathematics classes in which I feel something similar occurs.

David Cain is working with a group of twelve-year-olds and the task he has set them is to go from the net of a cube, with which they are familiar to one for a solid (which he terms a 'slanty-cube') where each face is a rhombus. The lesson extract opens with the word 'rhombus' written on the board and the net of a cube drawn. He says to the whole class:

Therefore, all you've got to do is this - very simple. There's the net of a cube - you've got to make these into rhombuses and it should stick together to make a ... slanty-cube. Yes, and instead of a cube, you'll have what is called a rhomboid - right?

Later in the lesson, he is talking with one particular girl, G, about why her paper model does not work. All of the language is quite inexplicit as both of their attentions are focused on the paper model throughout the conversation. [I have tried to give some feel for the overlapping nature of the conversation and also the relative duration of pauses. '...' means a short pause (up to half a second), longer pauses are marked with approximate durations.]

DC: This should fold up there.
G: No, it doesn't.
DC: But it don't, does it.

DC: Why, why doesn't it?
G That should be tilted that way.

DC: *That* one, *that* way. [checking] And then?
G: That way, and ... no, no. [She turns over piece of paper a couple of times. trying to see how it should go.]

Pause of several seconds

DC: laughs
G: [laughs, slightly ruefully] If I tilt them top three and then that bit ...
DC: Right.

G: ... Would that work?

pause (2 secs)
(**)
DC: What do you mean "Would that work?"? You're asking me. What do you think? You think I'm going to tell you?
G: No [laughs].
DC: No, all right then. [DC takes model]

pause of seven seconds, both looking at model

DC: [pointing at one particular fold] Is that alright?

(**)

G: Mm-hmm. [affirmative]
DC: Now what's the problem now - with that one?
G: That - that should be onto there.
DC: Yup - but it doesn't does it - you get a gap. OK, so what have you got to do?
G: Move that one to there.
DC: OK, you can try that. ...

I am very interested in the exchange marked between the sets of (**). David has not answered her question, in keeping with his desire to have pupils validate their work when they are able to. He does this by drawing her attention both to the question she has asked (by repeating it) and then by asking her whether she thinks it likely that he is going to answer it.

One interesting question is once you have raised the discussion to a meta-level, how do you then return it to the normal one where the form of utterances is not uppermost as the topic of attention. In this instance, David achieved this by means of a long pause and then offering a vary particular focus of attention back on one fold of the model and the question "Does this work?". Interestingly, the pupil did not turn the teaching gambit on its head by mirroring his earlier response and replying "What do you think? You think I am going to tell you?". This provides a nice instance of different rules applying to the teacher and the pupils - making meta-comments on the discourse is the prerogative of the teacher.

Moves to and from a meta-level need to be handled smoothly if they are not to be too disruptive of attention and focus, yet not so smooth as to pass unnoticed, as they are often important sites for teaching and, hence, potential learning.

In this second brief example, a second teacher (Dave Hewitt) is working with pupils at the board.

Dave: Uh-huh. Thank you very much. I've got another one.
[Writes 15 + 3 - 2 + 5 - 1 - 3 = 16]
But I'm not too sure whether I have done it right or not. Have I?

Pupil discussion: yes, no, no [more *nos*]
Dave: It's not quite right - OK, could someone make a change so that it is right. ... Thanks, Zena.

(**)
Zena: Can I just rub it out?

Dave: Yes, do. [With slight irony, as she has already rubbed out the final 3 with her finger and changed it to a 4.] You can even use a board rubber if you want to.
Zena: [Looks directly at Dave who is standing at the back of the class] Is that all right?

Pause (2 secs)

Dave: Zena asked a question.
[Chorus of yesses]
(**)

Dave: Could someone do that for me?
Pupil: [Speaking as he taps] That's 15, 16, 17 18, ...

Again I am particularly interested in the sequence marked by (**). Zena asked two questions that she may not have distinguished, yet which received quite different treatment from the teacher. The first one was a request for confirmation of procedure and permission - which was given - and the second was a request for verification, which evoked a meta-comment to the class, one to the effect that she had asked a question.

What sense can be made of the teacher commenting on something that everyone knows who was attending to Zena at the front of the class? Dave had ostensibly made a response, taken a turn in the conversation as the question had been addressed to him. But what he chose to do was to draw the class' attention to the form of the utterance as his response to it, rather than responding to it on the level at which it was made. Once again, the teacher himself had used those exact words a few lines earlier, indeed such a request for audience confirmation was built into the activity. (This meta-comment is not dissimilar to the way a counsellor or therapist might achieve a similar end, namely encouraging reflection on the nature of the speech act just made.)

Rosemary Clarke (1988), in her comparison of teaching and gestalt therapy, draws attention to the role of the therapist of attending to the process her client is engaged in by attending to the content - but not getting caught up in that content. In a similar way, the teacher must be able to stand outside the discourse as a commenter on it in order to teach. I am not proposing 'teaching as therapy' as another metaphor. Rather, I am suggesting that psychotherapy involves one particularly pure form of teaching, where the form of what is said is highly in focus by the therapist, who operates with a working belief that what is said *is* what is always meant. (This provides a reverse instance of the

metaphoric strategy outlines at the beginning, namely working with claims that X *is* teaching.)

If the teacher only makes meta-remarks, only answers questions with questions, the pupil may well lose confidence or trust in the teacher as a source of a conversation about the *content* they are struggling with, the teacher not being straightforward with them. Yet, if the teacher only engages with the content, there is the difficulty of 'teacher lust' (to use Mary Boole's evocative term - see Tahta, 1980, p.11), of the desire to tell the pupil things, which in part rescinds the possibility of the teacher teaching.

Conclusion

One important part of teaching mathematics can be seen as trying to inculcate communicative competence with respect to the mathematics register. Stubbs claims (1980, p. 115): "A general principle in teaching any kind of communicative competence, spoken or written, is that the speaking, listening, writing or reading should have some genuine communicative purpose". Yet this is at odds with my view that the classroom is an avowedly un-natural, artificial setting, in which the structure and organisation of the discourse by the teacher has some quite unusual features.

Teachers, in order to teach, need to acquire linguistic strategies (which I elsewhere have referred to as 'gambits' - Pimm, 1987) in order to direct pupil attention to salient aspects of the discourse - or indeed the nature of that discourse - while still remaining in 'normal' communication with the pupil. In focusing on instances of meta-communication, I have tried to highlight specific instances of teaching which seem to me central to the teaching enterprise as a whole. (In my talk, I will present an analysis of more ostensibly content-focused teaching, in order to support my claim that such meta-communication is widespread.)

This is a familiar tension which lies at the heart of teaching. One general formulation (referred to there as 'the didactic tension') runs as follows (Mason, 1988, p. 33).

> The *more* explicit I am about the behaviour I wish my pupils to display, the more likely it is that they will display that behaviour without recourse to the understanding which the behaviour is meant to indicate; that is, the more they will take the *form* for the substance.

> The less explicit I am about my aims and expectations about the behaviour I wish my pupils to display, the less likely they are to notice what is (or might be) going on, the less likely they are to see the point, to encounter what was intended, or to realise what it was all about.

Teaching necessarily operates within the constraints of this tension, and the phenomenon of meta-communication provides one instance of how it is lived by teachers in practice.

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Bibliography

Ainley, J. (1988) 'Perceptions of teachers' questioning styles', *Proceedings of the 12th PME Conference*, Veszprem, Hungary, pp. 92-9.
Clarke, R. (1988) 'Gestalt therapy, educational processes and personal development', in Pimm D. (ed.) *Mathematics, Teachers and Children*, Hodder and Stoughton.
Edwards, D. and Mercer, N. (1988) *Common Knowledge*, Methuen.
Garfinkel, H. and Sacks, H. (1970) 'On formal structures of practical action', in J. McKinney and E. Tiryakian (eds), *Theoretical Sociology*, Appleton Century Crofts, pp. 337-366.
Kaufman, B. (1964) *Up the Down Staircase*, Avon.
Mason, J. (1988) What to do when you are stuck. *ME234 Using Mathematical Thinking*, Unit 3, Open University Press.
Pimm, D. (1987) *Speaking Mathematically: Communication in Mathematics Classrooms*, Routledge and Kegan Paul.
Richardson, E. (1967) *The Environment of Learning*, Nelson.
Sinclair, J. and Coulthard, M. (1975) *Towards an Analysis of Discourse*, Oxford University Press.
Stubbs, M. (1974) *Organizing classroom talk*, Occasional paper No 19, Centre for Research in the Educational Sciences, University of Edinburgh.
Stubbs, M. (1980) *Language and Literacy*, Routledge and Kegan Paul.
Stubbs, M. (1983) *Language, Schools and Classrooms*, Methuen.
Tahta D. (ed.) *A Boolean Anthology*, Association of Teachers of Mathematics.

FOLDING BACK: DYNAMICS IN THE GROWTH OF MATHEMATICAL UNDERSTANDING

This paper explores the concept of "folding back" which serves as an essential element in the growth of a person's mathematical understanding. Our theory of such growth, of which this concept is a key part, observes understanding as a dynamic whole in which there are 8 levels from primitive knowing to inventising. Folding back allows a person functioning at an outer level and faced with a challenge to return to an inner level of understanding in order to re-construct that understanding as a basis for new outer level understanding. An example of the work of a university student is analysed to illustrate this concept and to show the roles of interventions by teachers or others in this process. Further examples, with protocols, drawn from students of various ages in schools will be presented at the conference.

Current curriculum documents on mathematics teaching, conference proceedings, psychological and artificial intelligence literature all exhibit interest in learning and teaching with understanding. Characterizing understanding in a way which highlights its growth, and identifying pedagogical acts which sponsor it, however, represent continuing problems.

This is, perhaps, because, as suggested by Minsky in a 1989 interview, that unlike machines people appear to understand something such as mathematics in many ways at once. We illustrate this with Jim's response to the following question (Kieren, 1990).

"Who gets more, a person A who gets a far share of pizza when 2 pizzas are shared among 3 persons or a person B who gets a fair share when 5 pizzas are shared among 8 persons?

Jim's Drawings

Figure 1

Jim's response: A gets more because 1/3 of 1/2 is 1/6 and 1/2 + 1/6 = 3/6 + 1/6 = 4/6 = 2/3. 1/2+ 1/8 = 4/8 + 1/8 = 5/8. 2/3 is more than 5/8. Because 1/6 is bigger than 1/8, because there are fewer pieces.

This could be seen as evidence of relational as opposed to instrumental understanding (Skemp 1976, 1987). Other classifications might use labels such as intuitive, symbolic, concrete (Schroeder, 1987). Although Jim's response is not much like a computable production system, one could identify the characteristics of understanding coherence and connectedness (Greeno 1978, Ohlsson 1988) .While such analyses observe understanding as a product of acquisitions, we seek to follow Pirie (1988) and observe mathematical understanding as a dynamic growing whole. Unlike Herscovics and Bergeron's work (1988), our theory, first put forward at PME 1989 (Pirie and Kieren, 1989) and extended and elaborated here, is not directly tied to development. Our model allows us to look at a person understanding "in many ways at once" and at the dynamics of how that comes to be and how it grows over various periods of time.

The basic model for our theory has 8 embedded levels which may be potentially exhibited by a person understanding a piece of mathematics. These levels start with primitive knowing: what the observer assumes a person knows as she is observed starting in her

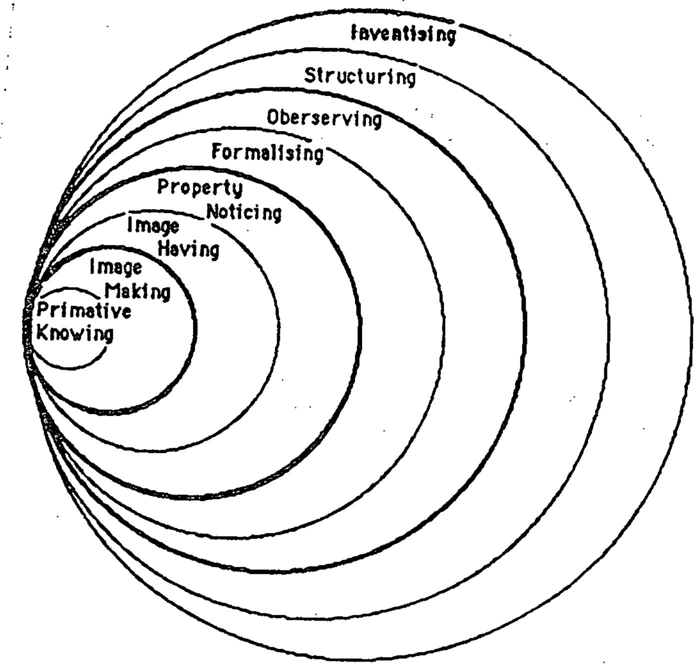

Figure 2

growth of a particular understanding. This understanding grows from singular image making activities to possessing and using mental images, to defining local properties, to more general formal mathematical ideas, upon which one can make formal observations which fit within a structure. The outermost level, inventising, is where, with full structural understanding of a piece of mathematics, one can deliberately ask questions that break out of this structure and create 'new' mathematics. These levels have been defined in detail elsewhere (Pirie and Kieren, 1989; Pirie and Kieren, 1990). It is important that the reader should be aware that the levels themselves do not constitute the understanding. They are named parts of a dynamic phenomenon which have no independent existence outside of the observation of a particular person's understanding of a particular piece of mathematics.

The rest of the paper focuses on a key dynamic concept in the theory. A person functioning at an outer level of understanding when challenged may invoke or **fold back** to inner, perhaps more specific local or intuitive understandings. This returned to inner level activity is not the same as the original activity at that level. It is now stimulated and guided by outer level knowing. The metaphor of folding back is intended to carry with it notions of superimposing ones current understanding on an earlier understanding, and the idea that understanding is somehow 'thicker' when inner levels are revisited. This folding back allows for the reconstruction and elaboration of inner level understanding to support and lead to new outer level understanding. Although we see growth in understanding moving from inner to outer levels, such that inner level knowing of some sort precedes outer level knowing, this process is not monotonic. The understanding process at any level always allows for and fosters folding back to go ahead.

Because we are trying to observe understanding as a dynamic growing process, we do not give tests with multiple items to check for the acquisition of mathematical understandings. Instead we observe and record students, working in small groups doing some particular mathematics over a period of time. so that we can analyse their dialogues as well as their written work.

ILLUSTRATIVE EXAMPLE

Richard was one of six university mathematics education students engaged for four hours in building a geometry for shapes created by a computer procedure that they controlled by inputting three parameters to generate one member of a potentially infinite set of geometric figures. In our previous publications we have used illustrative examples taken from school children. Richard was part of an experiment to look deliberately at older, mathematically more

mature students in order to observe the applicability of the model in such circumstances.

Richard and his partner Jean tried only a few examples before he said "Oh, they're just inward spirals" They tried a few more examples to generate and test the property that an "angle" input of 360/n generated an n sided polygon. While Jean now moved to look for other kinds of shapes, Richard stopped working on the computer He said "The program just generates spiral shapes by drawing a line of an input length, then turning right through the input angle. This is just repeated with the length reduced by an input decrement."

Considering Richard's work to this point, we note that after very few individual **image making** acts, he generated an **image** and then a few **properties** to elaborate it. His last remark suggests that he sees these shapes as a class controlled by a **formalised** statement. Richard then moved away to write-up his mathematics. He noted down the formal **observation** "the spirals generated by the angle 180-N and 180 + N are reflections of one another", and then set this observation in a mathematical **structure** by writing a short 'proof' based on his assumed formal procedural definition. In terms of the levels of the model above, we observe Richard moving quickly and directly from **image making** through the intermediate levels out to **structuring**

At this moment the teacher/researcher (TR) drew a square and asked "Could this be a member of your set?" Richard said "No". The TR then used the procedure to generate one on the screen. Richard returned to the machine and generated several examples trying to get a square. He was **folding back** to **image making**. Eventually he revised his **image** and his **formalisation** by saying - "Oh, I

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see, the decrement could be zero - of course the program doesn't stop".

In terms of the theory we might observe that Richard's understanding of spiralaterals had quickly grown "deep" - out to formal, structural levels. The TR's intervention invoked a folding back to inner level action. One might have thought that he would see the query on the square as a trivial consequence of his formalised understanding but he did not. He needed to reconstruct his understanding at an inner level.

Following this, Richard observed Jean working at the computer and producing stars. Once again Richard **folded back** to actual individual trials to try to generate parameters which would produce stars. He succeeded almost immediately and noted that some spirals had the property of being star-shaped. He then **folded back** again to image making and joined the hunt for the "angle" values which generated stars. This time the brief **folding back** to reconstruct his image was self-invoked on seeing other students' work. The return to seek clarification of those parameters which produced the star-shapes was self-invoked to deliberately enlarge his understanding of the newly noticed property of some members of the class of spirals.

We have tried to map the growth path of Richard's understanding in Figure 3. Oscillations in the lines are drawn to indicate some working before moving to another level of understanding. We trace the original growth of deep, but narrow understanding [solid and oscillating line (1)], the invocative interaction with the TR [at X], the initial folding back [dotted line (2)] and Richard's reconstruction of his image and noted additional property (3), now affected by his outer level formal

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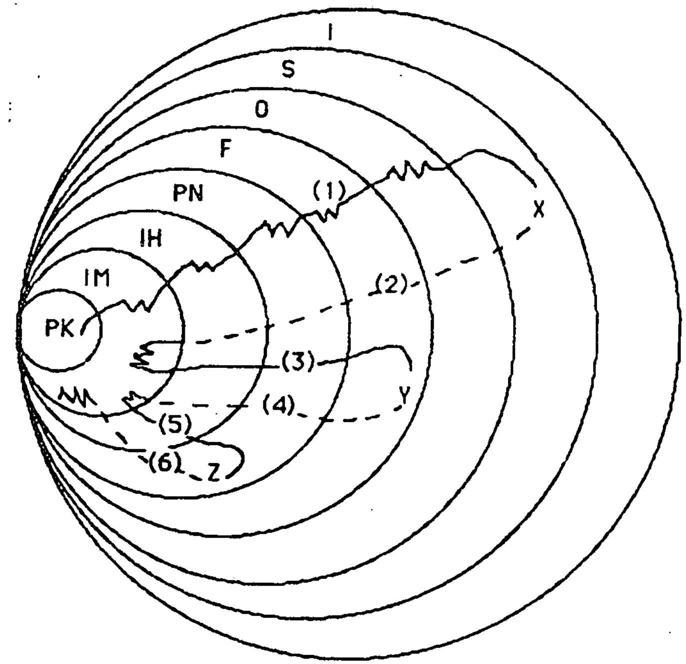

Figure 3

definition. The passive intervention (Y) involving folding back (4) and returning out (5) are drawn. Richard's attempts to elaborate his understanding at the property noticing level are shown by the folding back (6) to hunt for the 'special' angles. By repeated folding back Richard's understanding of spiralaterals had grown from a deep, but narrow character to include a broader understanding at a less formal level. There is of course more to the story of the growth of Richard's understanding than is illustrated here, but space does not permit further discussion.

<u>Summary</u>

Using our theory we have attempted to observe the growth of mathematical understanding by one student in one area of mathematics. Growth was seen from primitive knowing out to structuring but in this growth the inner level knowledge structures and actions were not "left behind". Richard folded back to

understand the mathematics in a more directed way. But this inner level knowledge building was stimulated by the more sophisticated understandings. Growth in mathematical understanding is not linear and monotonic. Understanding which is both deep and broad appears to require folding back to reconstruct bases for outer level knowing. This may not appear to be very "efficient", but if promoted by invocative interventions by teachers, fellow student and persons themselves, it does allow and account for the understanding of mathematics at many levels at once.

References

Greeno, J. (1978), Understanding and procedural knowledge in Mathematics Education.Educational Psychologist,vol.12(3), pp262-83.

Herscovics, N and Bergeron, J (1988), An extended model of understanding, in Proceedings of P.M.E.-N.A. 10, Dekalb, 111: Northern Illinois University, pp15-22.

Kieren, T.E. (1990), Understanding for Teaching Understanding, Alberta Journal of Educational Research, vol.36(3), pp191-202.

Minsky, M (1989), Artificial Intelligence, Discover, Oct. pp12-14.

Ohlsson, S (1988), Mathematical meaning and applicational meaning in the semantics of fractions and related concepts in, J. Hiebert and M. Behr (eds.), Number Concepts and Operations in the Middle Grades, Reston: N.C.T.M./L.E.A., pp53-91.

Pirie, S.E.B. (1988), Understanding - Instrumental, relational, formal, intuitive ... How can we know?, For the learning of mathematics, vol.8(3), pp2-6.

Pirie, S.E.B. and Kieren, T.E. (1989), A recursive theory of mathematical understanding, For the learning of mathematics, vol 9 (3), pp7-11.

Pirie, S.E.B. and Kieren, T.E. (1990), A theory of the growth of mathematical understanding - some elements and implications, a paper at the A.E.R.A. Meeting, Boston.

Schroeder, T.L. (1987), Students' understanding of mathematics: A review and synthesis of recent research, Proceedings of Psychology of Mathematics Education, XI, vol. 3. pp332-7.

Skemp, R (1976), Relational and Instrumental Understanding, Mathematics Teaching, vol. 77, pp20-6.

ENSEIGNER LES MATHEMATIQUES EN PREMIERE ANNEE SECONDAIRE APRES L'EVALUATION NATIONALE FRANCAISE.

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Résumé. *Nous analysons plus particulièrement la façon dont les professeurs de mathématique voient la gestion de leur enseignement de géométrie en première année de l'enseignement secondaire français. Tout au long de l'année scolaire, on a recueilli un corpus constitué d'entretiens avec 9 professeurs, de leurs propositions d'évaluation et de documents utilisés dans leurs classes. Au delà d'une belle unanimité des professeurs relative aux choix des contenus mathématiques précis de l'enseignement, l'analyse de ce corpus montre des manières très différentes de prendre en compte la variété des niveaux qui existent dans la manière dont se présentent les informations à l'entrée et à la sortie des traitements mathématiques. Cela va de l'ignorance de cette variété à son utilisation systématique, analogue à celle qui se trouve dans les niveaux de Van Hiele.*

Abstract. *We analyse the way the teachers see the organisation of their classroom instruction in geometry in the first year of the french secondary school. During one year, we collected a corpus consisting in interviews with 9 teachers, their propositions for evaluation and their choices of documents to use in the classroom. Concerning the choices of the content topics to be taught, the teachers are quite unanimous ; but the analysis of the already mentionned corpus shows great differences in the way the teachers take into account the variety in the levels of presentation of the informations at the entrance and at the exit in a mathematical treatment. Some teachers ignore this variety, some others use it systematically as shown in the presentation of the Van Hiele's levels.*

1. L'OPERATION NATIONALE D'EVALUATION ET NOTRE RECHERCHE.

En 1989, une évaluation de tous les élèves français a été faite à l'entrée de la troisième et de la sixième année de la scolarité (âges des élèves : environ 8 et 11 ans). Un numéro hors-série des cahiers "Education & Formations" (référence [MEN90] dans la bibliographie en fin d'article) précise les objectifs et les modalités de cette opération : ses deux volets principaux sont une évaluation des élèves, en mathématiques et en français, et une formation des professeurs pour leur permettre de *"procéder aux actions de soutien et de reprise d'apprentissage"* qui apparaissent nécessaires. Les résultats individuels de l'évaluation sont communiqués aux parents d'élèves. La même opération a été reprise à la rentrée scolaire 1990-91.

Nous avons procédé à une recherche conduite en mathématiques, au cours de l'année scolaire 1989-90, auprès des élèves de 22 classes de Sixième, dans 5 collèges, et de leurs professeurs. Les données recueillies auprès des élèves sont étudiées dans un autre document. La recherche

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entreprise auprès des professeurs a comporté le recueil et l'analyse d'un corpus susceptible de livrer des informations sur les moyens mis en oeuvre pour faire progresser les élèves selon leur point de départ en début de Sixième. Les entretiens et les questionnaires donnent des indications sur la *perception* que les professeurs ont de leur enseignement et des apprentissages des élèves. Le relevé de progressions d'enseignement et l'analyse des activités, des exercices, du cours, des évaluations, proposés aux élèves par nos collègues, donnent quelques indications sur la *réalité* de l'enseignement effectué.

Signalons que l'analyse est essentiellement centrée sur tout ce qui touche au domaine des travaux géométriques. L'acquisition de compétences en géométrie constituait particulièrement, de l'avis général des professeurs interviewés, un gros enjeu de l'enseignement en sixième. Cette belle unanimité sur l'importance de l'enjeu ne pouvait-elle pas masquer des formes d'enseignement nettement différentes ?

2. INSTRUMENTS PROPOSES PAR LES PROFESSEURS POUR EVALUER LES PROGRESSIONS DES ELEVES.

2.1. Définition du corpus et méthode d'analyse des instruments d'évaluation.

Au cours du deuxième trimestre nous avons invité chaque professeur à faire une proposition de test final. L'objectif de ce test était communiqué aux professeurs sous la forme suivante :

« Il s'agit d'élaborer une proposition de questionnaire destiné à tester tes élèves au sujet de l'enseignement que tu as mené jusque là (fin du deuxième trimestre), tant dans le domaine des travaux géométriques que dans le domaine des travaux géométriques. Le test doit être conçu pour deux séances de 50 minutes environ. Son objectif est de faire non seulement le point sur les connaissances et savoir-faire acquis depuis le début en sixième, mais aussi de mesurer les évolutions importantes à tes yeux depuis le début de l'année.»

Pour analyser les productions des professeurs, nous avons considéré les tâches que les élèves ont à effectuer comme consistant à opérer des transformations d'informations.

Nous avons alors considéré trois aspects dans ces transformations :

- Les *contenus mathématiques* évoqués ou à évoquer.
- La *complexité des transformations* à réaliser.
- Les *registres* dans lesquels s'effectuent *l'entrée et la sortie* des informations.

Précisons comment ont été pris en compte chacun de ces aspects.

Les contenus mathématiques : nous avons simplement dressé la liste des objets évoqués ou à évoquer (droites, droites parallèles, angles, triangles, triangles particuliers, etc.).

La complexité des transformations d'informations à opérer : nous nous sommes inspirés des niveaux de Van Hiele. En analysant les épreuves proposées par nos collègues aux élèves de Sixième, nous avons été amené nous même à distinguer 7 niveaux fins que nous avons finalement regroupé en trois niveaux qui correspondent approximativement aux niveaux 1, 2 et 3 de Van Hiele. En voici les définitions.

- Un *premier niveau de complexité* est celui où les informations sont simplement à transposer, néanmoins en coordonnant ou en enchaînant éventuellement certaines informations (ainsi « tracer la droite perpendiculaire à la droite d et passant par M »).
- Un *deuxième niveau de complexité* est celui où les informations sont soit à développer en remplaçant le défini par sa définition (« tracer un triangle ABC *isocèle* en A »), soit à concilier (« tracer un triangle ABC tel que AB = 5cm, AC = 8cm et BC = 6cm »), soit à condenser (« sachant que AB = 5cm, BC = 5cm et que les segments AB et BC sont perpendiculaires, que peut-on dire du triangle ABC ? »). Nous plaçons aussi à ce niveau les situations cumulant plusieurs des caractéristiques présentées.
- Un *troisième niveau de complexité* est constitué par les situations dans lesquelles, à partir des informations données, apparaît une production de nouvelles informations uniquement justifiée d'un point de vue mathématique par l'emploi d'une propriété. Nous ne distinguerons pas ici les cas où une justification est demandée de ceux où il ne s'agit que de conjecturer.

Les registres d'entrée et de sortie des informations : pour atteindre ce qui est considéré comme un but principal de l'apprentissage en géométrie et qui est en gros décrit par notre troisième niveau, les passages par les niveaux un et deux nous semblent nécessaires, mais cette maîtrise ne serait pas une condition suffisante. Pour reprendre certaines conclusions de R. Duval, il s'agit aussi d'*"apprendre à articuler plusieurs registres de présentation de l'information"*. Les enseignants se donnent-ils les moyens d'évaluer l'aisance avec laquelle les élèves changent de registres ?

Nous distinguerons ainsi les exercices par les registres dans lesquels on donne les informations aux élèves et les registres dans lesquels l'élève est amené à produire des informations. Le fait que TEXTE et FIGURE puissent être présents à l'entrée et à la sortie d'informations conduit à neuf types différents d'exercices. Une analyse plus fouillée s'appuie sur les formes que prennent les informations à l'intérieur de textes (langage naturel, langage symbolique, etc.) et de figures (mesures, codage, etc.).

2.2. Observations sur les questions proposées par les professeurs.

En ce qui concerne les CONTENUS MATHEMATIQUES évoqués dans ces épreuves nous remarquons une assez grande convergence : points, droites, segments et relations entre ces objets, triangles et triangles particuliers constituent le gros des contenus abordés par tous.

Nous pouvons néanmoins signaler que Cl-Wi-Gé (trois collègues travaillant en équipe et ayant élaboré une proposition commune), ainsi que Danièle, se distinguent en abordant les quadrilatères particuliers avec leurs propriétés (ce qui n'est pas au programme de la classe de Sixième).

Les angles ne sont pas évoqués par tous. Cela s'explique par le fait que les tests ont été élaborés au cours du deuxième trimestre et que les contenus dépendaient alors de la progression des classes à ce moment là. Ainsi, ne voyons nous apparaître ni symétrie orthogonale, ni géométrie

dans l'espace. Il est vrai aussi que le test devait se situer en grande partie dans le prolongement du test national du premier trimestre, pour repérer la progression des élèves.
A part quelques nuances, ce n'est donc pas sur les contenus que les tests se différencient.

NIVEAUX DE COMPLEXITE ET DIVERSITES DES ARTICULATIONS ENTRE REGISTRES

Tableau 1

	Ev. Nat. 1er trim.		Jean		Michel.	
	Trad	Extr.	Trad	Extr.	Trad.	Extr.
Niveau 1		1		2	1	
Niveau 2	1	2	1	1	1	
Niveau 3						1
Nombre d'articulations différentes par colonne	1	2	1	2	1	1
Nombre d'articulations différentes pour le test	3		2		2	

	Bernadette		Cl-Wi-Gé		Richard	
	Trad	Extr.	Trad.	Extr.	Trad.	Extr.
Niveau 1	1	2	1	1	1	1
Niveau 2	1	1	1	2	3	1
Niveau 3		2		1		
Nombre d'articulations différentes par colonne	1	3	1	2	3	2
Nombre d'articulations différentes pour le test	4		2		5	

	Joëlle		Danièle		Ev Fin 3è trim.	
	Trad.	Extr.	Trad.	Extr.	Trad.	Extr.
Niveau 1	1	2	2	2	3	2
Niveau 2	2	2	3		1	
Niveau 3		2				
Nombre d'articulations différentes par colonne	3	2	3	2	3	2
Nombre d'articulations différentes pour le test	5		4		5	

EXPLICATIONS DU TABLEAU 1 :
-Le pseudonyme de chaque groupé ou auteur de proposition de test figure au-dessus du tableau.
-Chaque test est ainsi schématisé par un tableau (3x2) de trois lignes correspondant aux trois niveaux et de deux colonnes, dont la première est réservée aux exercices qui sont des *traductions* intégrales d'une situation d'un registre dans un autre, et la deuxième aux *extractions* d'informations d'une situation.
-Dans chaque case figure non pas le nombre d'exercices, mais le nombre d'articulations de registres différentes
-Enfin sous chaque tableau figure le nombre d'articulations différentes de registres par colonnes et pour tout le test..

COMMENTAIRES :
Alors que les tests ne différaient pas grandement par les contenus, nous voyons ici des profils s'opposer.
La *diversité des articulations entre registres* sépare nettement les tests proposés en deux familles.
Nous avons d'un côté les tests qui se contentent de deux types d'articulations. D'un autre côté nous trouvons les tests qui proposent trois modes d'articulations en traduction de situations : à côté du décodage d'un texte pour construire une figure et de la reproduction d'une figure, nous trouvons aussi un travail d'encodage de programme de construction. C'est à dire qu'ici les principales situations de changement de registres sont systématiquement mises à l'épreuve. Se trouvent dans ce cas le test final du troisième trimestre et les tests de Danièle, Joëlle et Richard.
Du point de vue des *niveaux de complexité des exercices* proposés, nous distinguons trois profils, selon les proportions avec lesquelles apparaissent ces différents niveaux : niveau 1 principalement et niveau 2, niveau 2 principalement, les trois niveaux.

Tableau 2

	Diversité des articulations entre registres	Uniformité des articulations entre registres
Forte représentation du niveau 1 et absence du niveau 3	Ev Fin 3ème tri. Danièle	
Forte représentation du niveau 2 et absence du niveau 3	Richard	Jean Ev Nat 1er tri
Représentation équilibrée des trois niveaux	Joëlle	Bernadette Michel Cl-Wi-Gé

Nous pouvons alors opposer le groupe d'épreuves proposées par Danièle, Richard et l'évaluation du troisième trimestre au groupe d'épreuves proposées par Bernadette, Michel et Cl-Wi-Gé. Le premier groupe évalue les principales compétences de base en géométrie. Le deuxième groupe ose proposer à ses élèves des exercices de niveau 3 mais n'explore que très peu de changements de registres.
Restent en dehors de ces oppositions Joëlle d'une part, Jean et l'évaluation nationale du premier trimestre d'autre part. Pour Jean et pour l'évaluation nationale nous remarquons que l'évaluation de la lecture et celle de la rédaction d'un texte sont quasiment absentes. Phénomène de régulation ? Joëlle au contraire nous présente le test le plus complet. Tous les degrés de complexité, tous les changements de registres ainsi que tous les contenus sont scrupuleusement proposés à évaluation. Il est vrai que ce test se distingue aussi par le fait qu'il nous semble être le seul à ne pas tenir compte de la contrainte horaire figurant dans le cahier de charges. Joëlle le signale d'ailleurs en remarque vers la fin de sa proposition. Reste alors à savoir quels choix elle aurait fait pour respecter la contrainte de temps.

2.3. Repères utilisés et évolutions prévues et constatées chez les élèves par les professeurs.

Au premier trimestre nous avions eu un entretien avec les professeurs, au sujet des indications que donnaient le test de début d'année. Nous leur avions, à partir de là aussi demandé de faire des pronostics concernant les évolutions de leurs élèves. Nous avons renouvelé cette prise d'informations par rapport au test final. Mais cette fois ci nous l'avons fait sous forme de questionnaire dont le but proposé était de "permettre de produire des éléments de diagnostics en fin de sixième après la passation du test en mai". Ce questionnaire a été élaboré à partir de l'analyse de contenu que nous avions réalisé au premier trimestre. Il était donc demandé item par item du test final de:

- décrire les objectifs d'évaluation
- repérer les difficultés au vu des productions
- situer le degré de réussite pour la classe
- juger et expliquer le degré de gravité de l'échec d'un élève en fin de 6ème
- décrire et expliquer les évolutions depuis le début de l'année
- faire et justifier des prévisions d'évolutions pour l'avenir .

Au besoin nous avons demandé oralement, quelques explications supplémentaires au sujet de réponses trop laconiques.
Nous avons comparé les réponses selon deux dimensions:

1° LA NATURE DES REPERES que se donnent les professeurs pour décrire les objectifs des items et les productions des élèves. On peut opposer:

les REPERES QUI FONT REFERENCES A DES CONTENUS à connaître ou des savoir faire relatifs à ces contenus,

aux REPERES QUI FONT REFERENCES A DES COMPETENCES DEPENDANTES DES MANIPULATIONS DE REGISTRES.
Ainsi par exemple, "confondre parallèles et perpendiculaires", "ne pas connaître le vocabulaire relatif aux triangles particuliers" seront des diagnostics qui font références aux contenus. Par contre, le fait de "décoder une figure", de "ne pas savoir lire une phrase où il y a de nombreuses informations" sont des diagnostics qui font références à des capacités de maniements de registres. Nous avons par ces indices une indication sur le type de repérage qu'utilisent les enseignants. Mais ces repères ne correspondent pas forcément à des objets d'enseignement. Pour avoir une idée de cette correspondance il faut se reporter aux types d'explications que donnent les professeurs pour expliquer les phénomènes qu'ils observent.

2° LE TYPE D'EXPLICATIONS que donnent les professeurs pour justifier des résultats, des évolutions et des prévisions. Dans chacune des deux catégories de repères précédemment décrites nous discernons trois types d'explications :

– on fait confiance ou on incrimine une EVOLUTION ou une DISPOSITION NATURELLE indépendante d'un effort d'enseignement en mathématiques

– ou bien on évoque la présence ou l'absence d'EFFORTS D'ENSEIGNEMENT du professeur en mathématiques

– enfin on peut aussi invoquer les EFFORTS D'APPRENTISSAGE (suffisants ou insuffisants) des élèves.

Ainsi "l'échec par manque de maturité", "les difficultés en français" signalent des phénomènes indépendants des efforts de maîtrise des enseignants de mathématiques. En revanche, le fait qu'on "reverra la notion " ou qu'on "travaillera encore la production de programmes de construction" signalent des efforts d'enseignement. Enfin quand on dit que "les élèves n'apprennent pas assez leurs leçons" on renvoie nettement la balle dans le camp des élèves.
Au premier trimestre, nous avions constaté que d'un enseignant à l'autre, les analyses variaient en profondeur et en direction par rapport aux mêmes épreuves et erreurs. Il se produit effectivement une certaine homogénéisation sur le type de repérage effectué. Néanmoins, dans le détail, nous voyons encore d'importantes divergences concernant l'analyse de certaines erreurs. Si l'analyse faites par nos professeurs a de manière générale gagné en étendue et en profondeur, les oppositions restent par contre importantes sur l'interprétation des évolutions et les justifications concernant des pronostics pour l'avenir.
D'un côté nous avons les professeurs pour qui, les compétences dépendantes des manipulations de registres sont en cours d'acquisition, une acquisition étayée par des efforts d'enseignements nettement signalés. Pour donner un exemple, Joëlle signale que les progrès ont été importants en ce qui concerne la lecture et l'exécution de programmes simples (jusqu'à l'étape 4 pour le programme de notre évaluation) mais qu'il y aura des progrès aussi dans avenir pour des programmes où il s'agit de savoir prendre en compte plusieurs items à la fois et ces progrès se

feront grâce à un entraînement et aussi à une plus grande maturité. Elle rejoint à ce sujet Jean qui signale que cela devrait s'arranger en Cinquième par un travail spécifique.

De l'autre côté nous avons des professeurs qui restent plus volontiers dans le vague quant à la description des compétences en question et qui renvoient par exemple à des difficultés d'expression en français, irrémédiables pour certains élèves, ou pour lesquelles le temps ou l'enseignement en français arrangeront les choses pour d'autres. Il est à signaler aussi que pour ces enseignants la responsabilité d'apprentissage des élèves est plus souvent évoquée que pour le groupe précédent ("ils n'apprennent pas leurs leçons").

Les professeurs qui osent proposer en évaluation à leurs élèves des exercices d'une importante complexité de contenu mais n'explorent que très peu de changements de registres laisseraient effectivement plus facilement aux mains du destin, du hasard, de l'élève lui-même ou du professeur de français, le développement de compétences dans la manipulation de registres. A l'opposé, les professeurs qui proposent en évaluation les principaux changements de registre et ne s'aventurent pas encore dans l'évaluation de compétences trop complexes (embryons de démonstration par exemple), sont ceux qui explicitent le plus nettement des efforts d'enseignement dans les domaines évalués.

Si la conclusion précédente se confirme, on pourra ainsi constater que la proposition de tests étoffés avec leurs grilles de codage peut effectivement susciter des diagnostics plus fins, mais que ce diagnostic ne s'associera pas automatiquement à des pratiques en concordance. Le diagnostic ne sera pas suivi d'effet. Un champ de formation nécessaire s'esquisse ainsi.

Références bibliographiques

[MEN90] Ministère de l'Education Nationale, de la Jeunesse et des Sports, *Evaluation CE2-6ème : Résultats Nationaux*, numéro hors-série février 1990, D.E.P., 142 rue du Bac, 75007 PARIS

[PRA90] François Pluvinage et Jean-Claude Rauscher, *Les mathématiques en Sixième après l'évaluation nationale*, rapport de janvier 1990, à paraître dans les publications du MENJS

[PLU90] François Pluvinage, *Les mathématiques en Sixième après l'évaluation nationale*, rapport d'octobre 1990, à paraître dans les publications du MENJS

[DUV88] Raymond Duval, Ecarts Sémantiques et Cohérence Mathématique, *Annales de Didactique et de Sciences Cognitives*, vol. 1, 1988, IREM Strasbourg

[HOF83] A. Hoffer,Van Hiele based research ,in *Acquisition of mathematics concepts and processes*, Lesh et Landau edit., 1983, Academic Press, New York

[JGU89] A. Jaime et A. Gutiérrez, The learning of plane isometries from the viewpoint of Van Hiele model, *Psychology of mathematics education, Actes de la 13ème conférence internationale*, 1989, Paris

ÉTUDE DES MODÈLES IMPLICITES MIS EN OEUVRE PAR LES ENFANTS LORS DE LA RÉSOLUTION DE PROBLÈMES COMPLEXES METTANT EN JEU UNE RECONSTRUCTION D'UNE TRANSFORMATION ARITHMÉTIQUE.

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Several studies have shown the difficulties of transformation problems for primary and even high school students. The aim of our study is to get a better understanding of the implicit mental models used by children in complex arithmetic problems, involving the reconstruction of a transformation. The study was structured in two phases. First, we used a written test to identify different stable resolution patterns used by children (198 students, age: nine to twelve) in a set of complex transformation problems. Then, at each level (fourth, fifth and sixth grade), fifteen students, representing all patterns of resolution, were individually interviewed in order to make the different models used more explicit.

L'apprentissage des quatre opérations arithmétiques constitue un élément important du programme de mathématiques au primaire. Un des buts visés par cet apprentissage est d'amener l'enfant à comprendre chacune de ces opérations et à les reconnaître dans un contexte de résolution de problèmes. Mais cet apprentissage n'est pas sans poser de difficultés.

Nos observations réalisées dans le cadre de recherches antérieures menées par l'équipe Bednarz et Janvier nous ont amenés à nous intéresser plus particulièrement aux situations de structure additive mettant en jeu une transformation arithmétique et exigeant une reconstruction (Bednarz, Janvier et Poirier, 1983). Il en est ainsi par exemple du problème suivant «Marie a cinq billes. Son père lui en apporte d'autres; elle en a maintenant douze. Combien son père lui a-t-il apporté de billes?»De telles situations qui mettent en cause une transformation arithmétique font appel à un déroulement temporel où un état initial donné (dans l'exemple précédent: collection discrète) est transformé en un nouvel état de même nature dans lequel il est inclus. Dans ces situations, l'état initial et l'état final sont fournis et la question porte sur la transformation qui s'est opérée sur l'état initial. La transformation qui n'est pas explicitement donnée doit en quelque sorte être reconstruite par l'enfant; c'est ce que nous entendons par reconstruction d'une transformation arithmétique.

Plusieurs études (Conne, 1979; Carpenter et Moser, 1982; Vergnaud, 1982; Riley, Greeno et Heller, 1983; DeCorte et Verschaffel, 1985...) ont mis en évidence les difficultés que posent de telles situations pour les élèves du primaire dans les premières années de leur scolarité. Quelques études ont fait ressortir par ailleurs que cette difficulté n'est pas spécifique aux jeunes enfants mais se retrouve aussi chez des élèves plus âgés (Conne, 1979; Marthe, 1979; Vergnaud, 1982; Bednarz, Schmidt et Janvier, 1989...). Dans des situations plus complexes mettant en jeu une séquence de transformations, les enfants ont alors recours aux mêmes

...cédures erronées. Ces résultats portent à réfléchir. Pourquoi les mêmes erreurs se retrouvent-elles à différents niveaux du primaire (et même du secondaire)? Comment expliquer cette stabilité des erreurs et cette difficulté à se représenter dans une situation donnée une transformation arithmétique qui s'est opérée?

Plusieurs recherches actuelles en didactique des mathématiques et des sciences tentent de mieux saisir ce qui est sous-jacent à l'erreur, de mieux comprendre la pensée mathématique de l'enfant à travers les conceptions, les modèles implicites que celles-ci expriment. Notre recherche s'inscrit dans cette perspective. En examinant de plus près les procédures de résolution utilisées par des enfants (9 à 12 ans) dans différentes situations complexes impliquant une séquence de transformations où la reconstruction d'une transformation est en jeu, nous cherchons à découvrir sur quelles représentations mentales des relations entre les données celles-ci se basent.

Objectif

Notre étude vise avant tout à mettre en évidence les modèles mentaux implicites mis en action par les enfants dans des situations arithmétiques complexes mettant en jeu une composition de transformations, situations additives faisant appel à la reconstruction d'une transformation. Ces modèles peuvent être à la base des erreurs que les enfants commettent dans ces situations.

Méthodologie

Des élèves âgés de neuf à douze ans, ont été soumis à différents problèmes complexes faisant appel à une composition de transformations arithmétiques où la première transformation (T_1, $T_1>0$ ou $T_1<0$) et la transformation résultante (T_R, $T_R>0$ ou $T_R<0$) sont données dans l'énoncé du problème; la question est posée sur la deuxième tranformation. Les problèmes retenus, à des fins d'expérimentation, l'ont été sur la base d'une analyse conceptuelle préalable des situations additives faisant appel à une composition de transformations et à une reconstruction (Vergnaud, 1976). Les problèmes ont de plus été présentés (pour une même structure sous-jacente) dans deux contextes différents (transformations opérées sur des collections discrètes et déplacements impliquant des grandeurs continues); cette

introduction avait pour but de vérifier la stabilité des procédures utilisées dans des contextes différents, et donc des modèles implicites mis en oeuvre par les enfants face à une structure mettant en jeu la reconstruction d'un changement (transformation ou déplacement).

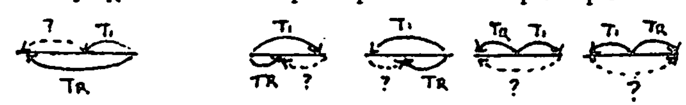
L'étude s'est faite en deux étapes. Une épreuve écrite, visant à mettre en évidence différents «patterns» de résolution stables utilisés par les élèves sur un ensemble de situations mettant en jeu la reconstruction d'une transformation, a été administrée à trois classes par niveau du deuxième cycle du primaire (enfants âgés de neuf à douze ans, pour un total de 198 sujets). (Le tableau 1 présente le profil du test écrit présenté aux trois niveaux scolaires).

Des entrevues individuelles ont été ensuite menées auprès de quelques enfants (15 enfants par niveau, représentant des classes de «patterns» de résolution), dans le but de rendre explicites les différents modèles utilisés par les enfants dans ces situations.

Tableau 1
Profil du test écrit aux trois niveaux

Variable inter-sujet Niveau scolaire	Variable intra-sujet Problèmes faisant appel à une reconstruction 5 types de complexité croissante Type 2	Type 3	Type 4	Type 5	Type 6
	2 contextes différents par type (transformation, déplacement)				
4e (3 classes)	- -	- -			
5e (3 classes)		- -	- -	- -	
6e (3 classes)		- -	- -	- -	- -

Séquence directe: $T_1.T_R$ même sens; $T_1.T_R<0$ — Séquence indirecte: $T_1.T_R$ même sens; $T_1>0$ $T_1<0$ — Séquence indirecte: $T_1.T_R$ sens opposé; $T_1>0$ $T_1<0$



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Analyse des résultats

L'analyse des résultats de l'épreuve écrite, met en évidence différentes classes de procédures stables de résolution erronées, où l'élève traite les transformations comme des états, les combinant de diverses façons. Ces résultats rejoignent les observations mises en évidence par d'autres études (Vergnaud, Conne, Bednarz et al...). Le tableau 2 donne en exemple la distribution des diverses procédures de résolution aux problèmes du test écrit pour la cinquième année (10 et 11 ans).

Tableau 2

Procédures des élèves de cinquième année au test écrit

Procédures de résolution		Structure des problèmes Type 3 A	Type 3 B	Type 4 A	Type 4 B	Type 5 A	Type 5 B	Type 6 A	Type 6 B
T_R	Transformation résultante	5	7	5	3	7	4	8	5
T_1+T_R	addition des deux données	7	12	8	4	2	1	1	2
$T_1\,T_R\,X$	Transformation résultante venant agir sur T_1	4	8	12	15	22	26	17	24
Besoin Éi	besoin d'un état initial pour résoudre le problème	3	0	4	1	5	1	4	1
Éi X Éf	comparaison de deux états	2	7	7	12	20	20	19	19
Réussite	bonne réponse	39	27	23	26	4	4	11	7
Autre		1	2	4	3	2	5	3	3
Aucune réponse		4	2	2	1	3	4	2	4

Total: 65 sujets

A: contexte transformation / collection discrète
B: contexte déplacement / grandeur continue

Une analyse des protocoles d'entrevues nous permet par ailleurs de mieux cerner le fonctionnement des différents modèles implicites guidant les actions des enfants dans la résolution de ces situations. Cette analyse indique que certaines procédures erronnées identifiées au test écrit relèvent d'une même stratégie, d'un même modèle mental. Trois grands modèles ont ainsi été dégagés:

1- Dans un premier modèle, que nous qualifions de «linéaire», l'enfant traite la première transformation comme un état initial sur lequel vient agir la transformation résultante. Le sujet linéarise donc l'information fournie dans le problème ne voyant pas qu'il y a une reconstruction à faire, et transforme à cette fin les données du problème pour se ramener à une structure directe du type E_i(État initial) + (transformation)?

2- Dans un deuxième modèle, que nous qualifions de «comparaison» les transformations sont traitées à nouveau comme des états par l'enfant, mais le sujet voyant qu'il y a une reconstruction à faire, compare ces deux états pour en trouver la différence.

3- Dans un troisième modèle, conduisent alors à une réussite, le sujet travaille en termes de transformations (et non d'états), et perçoit la reconstruction à effectuer. Ce modèle, où les données sont effectivement traitées par l'enfant comme des changements, demeure toutefois minoritaire, moins de 10% des sujets y auront recours.

Bibliographie

BEDNARZ, N., JANVIER, B., POIRIER, L. (1983). Problèmes de représentation d'une transformation en arithmétique chez les enfants du primaire, *Bulletin de l'Association Mathématique du Québec*, octobre.

BEDNARZ, N., SCHMIDT, S., JANVIER, B. (1989). *Problème de reconstruction d'une transformation arithmétique*. Rapport de recherche FCAR, CIRADE, UQAM

CARPENTER, T.P., MOSER, J.M. (1982). *Addition and Subtraction. A Cognitive Approach.* Hillsdale, N.J., Lawrence Erlbaum Associates.

CONNE, F. (1979). Pierre, Bertrand, Claude, Paul, Laurent, Michel et leurs billes. Contribution à l'analyse d'activités mathématiques en situation, *Approches en psychopédagogie des mathématiques*, Université de Genève, 25-84.

DECORTE, E., VERSCHAFFEL, L. (1985). Influence of Rewording Verbal Problems on Children's Problem Representations and Solutions. *Journal of Educational Psychology*, vol 77, no 4, 460-470.

MARTHE, P. (1979). Additive Problems and Directed Numbers, in *Proceedings of the annual meeting of the International Group for the Psychology of Mathematical Education*, Warwick, Angleterre.

RILEY, M.S., GREENO, J.G., HELLER, J.I. (1983). Development of Children's Problem-solving Ability in Arithmetic, *The Development of Mathematical Thinking*, ed., H. Ginsburg, New York.

VERGNAUD, G. (1982). A Classification of Cognitive Tasks and Operations of Thought Involved, in *Addition and Subtraction Problems, Addition and Subtraction. A Cognitive Perspective*, ed. T.P. Carpenter, J. M. Moser, T. Romberg, Hillsdale, N.J., Lawrence Erlbaum Associates.

CLASSROOM ASPECTS WHICH INFLUENCE USE OF VISUAL IMAGERY IN HIGH SCHOOL MATHEMATICS.

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Research data suggest that some classroom aspects are more facilitative of the effective use of imagery in high school mathematics than are others. In the study described, teachers' beliefs about imagery use in high school mathematics resulted in classroom aspects which affected visualisers' learning. Teachers in three groups, visual, middle and nonvisual, were compared. Some middle group aspects appeared to be optimal, but attitudes and actions of teachers in all groups created subtle learning effects. A second phase of the ongoing study, in 1991, looks at classroom aspects which help low achieving visualizers

A note concerning the radical constructivists' viewpoint.

In this paper I shall refer to the **findings** of the study, a term whose use is not consonant with the philosophical position of radical con structivism, since it may suggest an objectivist position (Tobin, 1990) The study I describe is interpretive (Eisenhart, 1988), and I subscribe to the view that the events described may admit of other interpreta tions. As Goethe wrote, "Es hort doch jeder nur was er versteht" ("Each person hears only what he understands"). My rationale, then, for using the word "findings" is that in the research described what emerged was so surprising to me, the researcher, that I scarcely believed it and therefore have not submitted it for publication, beyond a mere mention, in the six years which have elapsed since the study. My interpretation was informed by an earlier, intensive study of Albert Einstein's creativity (Presmeg, 1981) which enabled me to recognise elements and issues relating to creativity as these cropped up later, in the research on visual imagery in the classroom. The mathematics education community is more aware now than in 1984, when Thompson's paper appeared, of the importance of teacher beliefs Jakubowski, 1990) and of the role of "products of the imagination such as metaphor, metonymy, and mental imagery" (Lakoff, 1987, p. 165) in the classroom. Accordingly I offer the following interpretation of the influence of teachers' beliefs and attitudes about imagery in class room aspects which, in turn, influence the learning of mathematics by visualisers. (For definitions of terms such as "visual image", "visua liser", "teaching visuality" and "visual teacher" see Presmeg, 1986b

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Classroom aspects (CAs) from the literature.

A comprehensive survey of the literature (mainly psychological) up to that time (1984) suggested that a classroom facilitative of effective use of imagery in learning high school mathematics would include the following aspects. (References and a full rationale are given in Presmeg, 1985, pp. 126-133.) In the fieldwork it emerged that all but one of these CAs (teacher use of gesture, which was nevertheless useful as an indicator of imagery) were helpful to a majority of those of the 54 visualisers in the study who experienced them.

1. A general classroom atmosphere or ethos which is controlled but relaxed and unhurried. These three aspects were examined separately.

2. Pictorial presentation by the teacher of mathematical topics. A diagram in mathematics is often essential to a topic (especially but not exclusively in geometry). In this CA, therefore, more weight was given to diagrams which were inessential in the sense that the mathematics could be understood without them.

3. Teacher formation and use of a range of their own images. This CA includes
(a) use of number forms and algebraic imagery (numerical or algebraic images of a generic nature, "pure shapes", which may be conveyed by arrows, lines or a teacher's fingers), and
(b) use of gesture, by which a teacher's "talking" with his or her hands conveys an image to pupils.

4. Conscious teacher attempts to generate imagery in the minds of pupils. This CA includes
(a) use of instructions to pupils to form images (which would have to be controllable to be useful), and
(b) creation of a dynamic situation in which pupils think in moving pictures. This dynamism overcomes the limitations of "frozen" pictures, which may constitute a serious hindrance in mathematical thinking.

5. Teacher use of the motor component of pupils' imagery. This CA includes
(a) pupil use of arm, finger or body movements, and
(b) use of concrete materials (structural apparatus or manipulatives).

6. Use of colour
(a) by teachers,
(b) by pupils, encouraged by their teacher.

7. Teaching which is not rule-bound. This CA includes
(a) teacher use of intuition, and encouragement of pupils to use intuition,
(b) use of pattern-seeking methods,
(c) delayed use of symbolism,
(d) deliberate creation of cognitive conflict in pupils, and
(e) demonstration or acceptance of alternative methods.

Triangulation of viewpoints (teacher's, pupils' and researcher's) gave an indication of which CAs were used by individual teachers. In an item analysis, four of these seventeen CAs did not discriminate well between visual and nonvisual teachers, viz., all three of the classroom atmosphere aspects, and the last CA, teacher use of alternative methods, which was judged to have been used by all teachers. (However, there were differences in attitudes to alternative methods - see later.) If these four aspects and pupil use of colour, which appeared to relate more to pupil needs than to teacher presentation, were excluded from the CAs, then the teaching visuality (TV) scores using the remaining twelve aspects divided the thirteen teachers very neatly into three groups, i.e., a visual group (5 teachers), a middle group (4 teachers) and a nonvisual group (4 teachers). (Spearman r for the rank orders based on 12 and 17 CAs was 0.961, significant $p<0.01$, $N=13$.)

CAs which were particularly effective in distinguishing the groups were
7(b) use of pattern-seeking methods (where 0, 2, 5 were numbers of teachers who used the CA in nonvisual, middle and visual groups respectively),
3(b) use of gesture (1, 4, 5)
5(b) use of concrete materials (0, 0, 4)
7(c) delayed use of symbolism (0, 2, 4), and
7(d) deliberate creation of cognitive conflict (1, 3, 5).

In contrast, two CAs were used by very few teachers, viz.,
4(b) use of moving imagery (0, 0, 1) and
5(a) pupil use of arm, finger or body movements (0, 1, 1).

All of the above findings, while interesting, were not particularly startling. What were startling were the unsought CAs which emerged in eight months of classroom observation, and the regularities in beliefs and attitudes which underlay them, as follows.

Classroom aspects characteristic of three groups of teachers.

Transcripts of lessons were analysed to identify aspects characteristic of the groups of teachers in four areas, viz., aspects concerning teaching, pupils, mathematics and visual methods. In the analysis, the overarching and immediately noticeable distinguishing characteristics of lessons with teachers in the visual group were their fullness and variety, in striking contrast to the characteristics of lessons with teachers in the nonvisual group. It is not that the visual teachers and the nonvisual teachers used sets of aspects which did not intersect: rather, the aspects used by the nonvisual teachers comprised a very small subset of the large set of aspects used by the visual teachers. The set of aspects used by the middle group of teachers was intermediate between the other two sets. Twelve of the thirteen teachers used a lecturing style at times in the 108 lessons observed. This aspect, therefore, did not differentiate between groups. However, lecturing was the predominant mode of instruction for teachers in the nonvisual and middle groups, while it was used only occasionally by four of the five visual teachers.

The essence of the teaching of those in the visual group is captured in the word **connections**. The visual teachers constantly made connections between the subject matter and other areas of thought, such as other sections of the syllabus, other subjects, work done previously, aspects of the subject matter beyond the syllabus, and above all, the real world. In addition to these cognitive links or bridges, the visual teachers used many different ways of presenting their subject matter, and they were far more inclined to elicit different methods of solution of problems from their pupils, whose diversity they recognised. These aspects naturally resulted in increased use of pattern-seeking methods and cognitive conflict questions, in fact of all the CAs subsumed in the category, "teaching which is not rule-bound".

It must be stressed that no value-judgement is implicit in these findings. Some of the teachers in the nonvisual group, as also in the middle and visual groups, were excellent teachers for certain of their pupils. The nonvisual teachers were simply more inclined to present the work formally, logically and rigorously from the start - i.e., in a convergent manner - than were the visual teachers, whose teaching may be described as divergent.

It was a totally unexpected finding that visual and nonvisual teachers were distinguishable in terms of certain characteristics associated with creativity. However, once this possibility was recognised, the evidence multiplied to suggest that the phenomenon is more far-reaching. All the classroom aspects used by visual teachers may be interpreted as stemming from a visual **cast of mind** which includes personality traits associated with creativity (Vernon, 1970), such as openness to external and internal experience, self-awareness, humour and playfulness, in addition to the cognitive divergence manifest in their tendency to make connections between areas of thought. Certainly humour and expression of their feelings characterised the lessons of the visual teachers in this study, and their self-awareness emerged in qualitative data obtained by multiple methods.

Attitudes and beliefs.

Teachers in the middle group often used aspects which were characteristic of the classrooms of visual teachers. What distinguished the two groups were their beliefs about use of imagery in mathematics, and attitudes which resulted from those beliefs. The following classroom aspects appeared to arise from these beliefs and attitudes. (T stands for teacher.)

	Number of teachers using CA (nonvisual, middle, visual)
T uses language evocative of imagery	(0, 2, 5)
T suggests that a visual presentation is useful	(0, 0, 4)
T expresses patterned visual thought	(0, 3, 3)
T suggests the importance of a visual interpretation	(0, 1, 3)
T suggests that visual means concrete or practical	(0, 0, 2)
T expresses a preference for visual methods	(0, 0, 2)

Four CAs were used only by teachers in the middle group, viz.,

T encourages remembrance of a generalised principle	(0, 4, 0)
T suggests that a visual interpretation is not really necessary	(0, 2, 0)
T suggests that a visual method is not interesting	(0, 1, 0)
T suggests limitations of a visual method	(0, 1, 0)

While teachers in the visual group were unanimously positive in their attitudes towards the visual methods which they used, teachers in the middle group typically conveyed to their pupils the impression that the visual method - which they might demonstrate - is not really necessary but an inferior way. These differences, only hints of which can be given here, emerged very clearly in the lesson transcripts, e.g. as follows.

MR BLUE (middle group): You don't need to do this [the diagram]. There would be no need whatsoever to do this. You can do it all by calculation. I only just put these down [i.e. diagrams] so you can see it.

When a boy in MR BLUE's class found an answer to a problem on arithmetic sequences by a visual means and asked if it were acceptable, MR BLUE replied, "I wouldn't mark it wrong, but I want simultaneous equations!"

For MR BLUE, the beauty of mathematics resided in its algebra.

MRS GOLD (visual group): So you feel that that's fine as an answer [i.e., vector components obtained by calculation]? Alright, could you now prove to me by drawing that that's true? Could you draw that? And don't forget, vectors, all the way down the line, will go to the geometry with the algebra. So this is in actual fact the algebra, which is representing some picture there. How are you going to have a look at that thing?

Nonvisual teachers typically avoided reference to visual methods completely.

The question naturally arises of what effects these attitudes and beliefs, as conveyed by means of the classroom aspects, might have on visualisers' learning of mathematics. Again, no value judgements are implied: effects on, say, nonvisualisers might be very different. Nonvisual teachers typically did not express positive or negative attitudes towards visual cognition. Feeling no need of visual supports themselves, they ignored these as far as possible, with the effect that visualisers in their classes attempted to dispense with their preferred visual methods. In the classes of nonvisual teachers, visualisers appeared to experience difficulties and dilemmas, and their attempts to adapt their preferred cognitive mode to be more in line with their teacher's met with varying degrees of success. Not one of the fourteen visualisers with teachers in this group was "successful" in the sense that they achieved highly in tests and examinations. Nonvisual teaching had the effect of leading visualisers to believe that success in mathematics depended on rote memorisation of rules and formulae. This finding is hardly surprising considering the teaching emphasis on formal logic, rigour and rules in this case. Those visualisers who achieved a limited measure of success (e.g. LAURA D with MRS CRIMSON) were cognitively very dependent on their teacher.

With teachers in the middle group, visualisers appeared to benefit from their teacher's stress on abstraction and generalisation. Pattern imagery and rapid use of curtailed methods were encouraged in the thinking of visualisers with these teachers, thereby overcoming some of the limitations associated with visual thinking in learning mathematics (Presmeg, 1986 a and b). Only two of the 54 visualisers achieved highly in their school-leaving mathematics examinations, and they were both in classes with middle group teachers. Of nine relatively high achievers, five were with teachers in this group. The importance, as in Krutetskii's (1976) structure, of generalisation, curtailment and logical economy for high mathematical achievement, was confirmed in this study. The teaching of middle group teachers was in this sense found to be optimal for these visualisers. However, no fieldwork pupil decided to continue with mathematics as a career choice, and the highest achievers with teachers in the middle group also manifested cognitive dependence on their teachers (e.g. CRISPIN T with MR BLUE).

The rich and varied classroom aspects of visual teachers were generally viewed very positively by visualisers. But the worlds (Bruner, 1976) of the classroom are seldom simple, and visualisers did not construe all aspects of these CAs as beneficial. In the visual group, MISS MAUVE was exceptionally self-aware, integrated and reflective in Thompson's (1984) sense. MISS MAUVE drew from her pupils many different methods of solution to mathematical problems, as might a constructivist teacher (Kamii, 1990). Her reply to the researcher's question about whether she puzzled her pupils on purpose (CA number 7(d)) is quoted in full to illustrate her perception of some of the interaction problems which arose.

MISS MAUVE: Mmm. Sometimes it happens on purpose. Sometimes it happens by accident [laughing]. But I do try to make them puzzle. .... Now I think, actually, that that is what children find the most difficult, maybe, in adjusting to my teaching. Um, sometimes people say to me, but you, you make them think [little laugh], when I haven't meant to do that. Now it doesn't always work well for them because sometimes, you know, I think a pupil might feel as though she's a worm on the end of a pin. Because I, I don't want to leave her alone until I feel she's got some idea of where I'm trying to take her. So I sort of tend to .... she asks me the question and I don't give her the answer, I may ask her another question; and then she answers that one and I ask her something else, and some of them find that, um, perhaps too threatening. Because they feel that, that I'm not just telling them what they want to know.

INTERVIEWER: Possibly this is the very way of teaching that will create mathematicians.

MISS MAUVE: It may do, but it also.... they have to have the emotional .... you know, ability to, security to handle it. And I think that sometimes I'm so.... so keen to see that intellectual stepping, that I forget that they can't handle it. So, I mean sometimes if they eventually dissolve into tears it comes as a shock. That they find this such an agonising process, you know. That I've got them muddled up; they feel as if I've got them muddled up.

With teachers in all groups (even the visual) there was a tendency for more highly achieving pupils to be cognitively dependent on their teacher. This interpretation led to the writer's conjecture that visual teachers might be more effective than middle group teachers in facilitating independence in learning mathematics, if they were more aware of potential pitfalls in the classroom aspects they used. If visualisers could be helped to experience successful independent learning, more of them might have a chance of entering the category of "stars" (Presmeg, 1986a), and more of them might enjoy mathematics for its own sake sufficiently to make it their career choice.

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References.

BRUNER, J. (1986) Actual minds, possible worlds. Cambridge, MA: Harvard University Press.

EISENHART, M.A. (1988) The ethnographic research tradition and mathematics education research. Journal for Research in Mathematics Education, 19(2), 99-114.

JAKUBOWSKI, E. (1990) Beliefs and metaphors associated with mathematics teaching and learning. Paper presented at the 68th Annual Meeting of the National Council of Teachers of Mathematics, Salt Lake City, Utah, April, 1990.

KAMII, C. (1990) Constructivism and beginning arithmetic. In Cooney, T.J. and Hirsch, C.R. (eds) Teaching and learning mathematics in the 1990s. Reston, Virginia: National Council of Teachers of Mathematics, 1990 Yearbook.

KRUTETSKII, V.A. (1976) The Psychology of mathematical abilities in schoolchildren. Chicago: University of Chicago Press.

LAKOFF,G. (1987) Women, fire and dangerous things: what categories reveal about the mind. Chicago: University of Chicago Press.

PRESMEG, N.C. (1981) Albert Einstein's thought, creativity and mathematical education. Unpublished M.Ed. dissertation, University of Natal.

PRESMEG, N.C. (1985) The role of visually mediated processes in high school mathematics: a classroom investigation. Unpublished Ph.D. dissertation, University of Cambridge.

PRESMEG, N.C. (1986a) Visualisation and mathematical giftedness. Educational Studies in Mathematics, 17(3), 297-311.

PRESMEG, N.C. (1986b) Visualisation in high school mathematics. For the Learning of Mathematics, 6(3), 42-46.

THOMPSON, A.G. (1984) The relationships of teachers' conceptions of mathematics and mathematical teaching to instructional practice. Educational Studies in Mathematics, 15, 105-127.

TOBIN, K.G. (1990) Personal Communication.

VERNON, P.E. (1970) Creativity: selected readings. Middlesex, England: Penguin.

COMPUTER ACTIVITIES IN MATHEMATICAL PROBLEM SOLVING WITH 11-14 YEAR OLD STUDENTS: THE CONDITIONAL STRUCTURE LEARNING

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SUMMARY

During a research activity into algorithmic aspects of arithmetic and solutions to simple problems of applied mathematics by computer with 11 to 14 year olds, some problems associated with programming activities in this age group became evident. The research study presented here has examined conditional structure through the results to certain problems given in a test, after about 25 hours of mathematics lessons with computers, half of which was carried out working on the computer itself. The tests were arranged to allow the examination of the different difficulties found by the students when confronted with problems which this structure poses both on a conceptual level and a programming language level. Answers are classified underlining different levels of comprehension of the conceptual scheme in question.

INTRODUCTION

Certain algorithmic aspects of arithmetic and applied mathematical problem solving with computers by 11 - 14 year old students, pose new learning problems which are directly associated with programming activities.

The research study presented here has concentrated, in particular, on the difficulties associated with the use of conditional structure, and was carried out among the students of certain classes involved in a project (created by the Didactic Research Group of Pavia), which includes the use of computers for normal mathematical teaching in lower secondary school classes. The objective of this project is to contribute to the students' mathematical training and to facilitate comprehension of conceptual schemes and their use in problem solving by the pupils, through programming with BASIC language (Reggiani, 1988).

The aim of the research study, of which the part treating the conditional structure is presented here, is to analyse separately the difficulties associated with the programming language use, and the difficulties associated with the algorithm construction, the concept of variable, and the selection and cycle structures. Although, because of the equipment available to us, it has happened that our work was carried out in BASIC language, with all the limitations that this language presents (Hoyles-Noss, 1987) it seems to us that useful elements can be found for comparing with research studies carried out in other contexts and in other programming environments (Samurçay, 1985; Capponi-Balacheff, 1989; Haussman-Reiss, 1989).

THEORETICAL FRAME

Problems associated with learning conditional structure in computer activities have been widely covered

by Rogalski (1987).

It should be remembered that in BASIC languages, this structure presents a condition generally expressed through an equality or an inequality; in the simpler BASIC languages, if the condition is verified the same line commands are carried out, otherwise the following line. Moreover, it is often necessary to also use jump instructions to refer to instruction blocks which must be carried out under the condition in question.

From a conceptual point of view (logical and informatical) the difficulty consists in mastering the alternative structure, and distinguishing, within the problem, what is or is not subject to the alternative.

Even more, from a strictly logical point of view, it is necessary to know how to negate a proposition.

It is useful to note (Pellerey, 1986, p. 107) that a fundamental difference between the selection structure in computer science and the logical implication lies in the fact that in computer science, the "choice" is determined by the truth value of the conditioned proposition in question, whereas in logic (and sometimes in common language) the same consequence can be derived from a true or false proposition. This observation opens up the possibility for research, comparing conceptual structures in computer science and in logic, which however, lies outside the limits of this study.

In this study, the conditional structure is seen as a conceptual scheme. Reference is mainly made to the meaning which is given to the term "schème" in Vergnaud (1985, 1988), as a behaviours' structure which remains invariable for a class of situations. In the solution of a problem, the subjects use certain rules which can even be represented graphically (schémas).

From this first comprehension mode, the pupils go on towards a more general understanding - the conceptual scheme - when they are capable of recognising the possibility of using this scheme in a similar or more complicated problem, and of making decisions in consequence. We could also say that, at the first level, they are limited to following a script, and then at the second level, they interpret the script as a significant whole, and not only as a simple sequence of behaviours - that is - they know how to carry out an action which demonstrates an implicit idea and which can be interpreted by the teacher as "concept in action" (Vergnaud).

So, when applying the scheme concept, perhaps it is possible to distinguish between two modes of the concept itself: that which is operative, connected to the particular programming language, and that which is conceptual, linked to the logical structure involved.

RESEARCH PROBLEM

The research problem under consideration here is that of classifying the difficulties associated with the comprehension of conditional structure.

Especially in work of error analysis and classification, and keeping in mind what has been said in the theoretical frame we set out to verify whether:

a) there are pupils who cannot even comprehend the operative scheme and who simply limit themselves to using the name. These pupils do not "see" the duplicity of the alternative.

b) there are pupils who can follow the language scheme without error, but who are not capable or transferring it to a different situation.

RESEARCH CONTEXT AND METHOD

The study presented here is carried out with the pupils of three classes - two third classes (13 year olds) and a second class (12 year olds) - which were considered as representative because of the variety of the social and cultural background of the pupils.

The pupils under consideration had followed a programme of about 25 to 30 hours of computer lessons, half of which, on the computer itself (part of the programme was carried out the previous year).

They were taught the first elements of programming through simple arithmetic problems. The instructions presented to them were INPUT, LET, PRINT, IF ... THEN, GOTO, FOR ... NEXT.

Above all, the instruction for alternatives was presented with very simple examples and also represented with flow charts, to try to supply a scheme which was adapted to the programming language: "If the condition is verified - proceed to the right, otherwise proceed underneath".

In the first examples, we tried to avoid using GOTO in order to concentrate attention on the fundamental structure.

At the end of this work stage, we proposed a test on all the subjects treated. This test included seven questions of which three were multiple choice questions, suitable to check their knowledge of the instructions and their capacity to write and decode simple programmes. One of these questions concerned with the instruction IF ... THEN... was deliberately worded so that this instruction played a central role with the possibility of underlining the different possible errors. With this instruction, in fact, an added sequence is established besides that which makes up the programme without the condition, thus creating more separate sequences.

The aim was to examine the pupils' behaviours in front of the sequences multiplicity.

The question was the following:

"On a school day-trip, the ticket costs 5,000 lire per pupil, and lunch costs 10,000 lire each if the pupils total 20 or less; and 8,000 lire each if the pupils total more than 20. Create a programme which can calculate the total cost of the trip according to the number of the pupils".

The results of the test showed about 20% of correct programmes as against an average of about 55% of correct answers to the other questions in the same test.

We carried out a detailed analysis of the protocols, subdividing the incorrect answers among the following different type examples:

A) transformation of the problem into "mono-sequential", writing - that is - of the programme without even using the alternative instructions;

B) writing of the programme with the instructions for the alternative but still thinking of it as "mono-sequential" (only for the case in which the condition is verified, the further instructions are written);

C) presence of syntax errors conneceted with the language used (for example: lack of GOTO) or errors connected with the text of the problem (confusion between the price of the meal and the price of the trip).

The excessive complication of the test where one element (price of the ticket) is not subject to the alternative, while the other (price of the meal) is subject, did not permit complete classification of the errors and an exact quantification of the types which resulted because of the interference of the problem described above.

Moreover, since it was a test given at the end of the school year, the summer holidays prevented the use of an interview with the pupils to confirm the attribution of the protocols to the type examples.

Therefore it was decided at the beginning of the new school year to propose to the same pupils a re-examination of their protocols and an implementation of their programmes on the computer. This could be considered meta-cognitive, in that it encourages the pupils to think about elements already learnt and can favour the transition over to generalisation and therefore to the understanding of the conceptual scheme (Concept in action - see theoretical frame).

After this activity, a new test was proposed which, with the previous analysis of the problem in mind, was to enable us to evaluate the level of "non-comprehension" of each individual pupil.

The method criterion used for processing the new questions, was that of isolating the conditional structure in three situations that could be compared to one another:

i) An already existant given programme with conditional structure. They were asked what effects it produces.

ii) An applied mathematical problem. They were asked for a solution in the form of a programme.

iii) A very simple question with a completely computer content. They were asked to write the programme.

The test they were given is as follows:

1. Consider the following programme:

```
10 input S
20 if S < 10 then P = 3*S : goto 40
30 P = 2*S
40 print P
50 end
```

a) What appears on the display when you give the command RUN? Why?

b) Can you introduce the values for S?

10? yes no

8? yes no

25? yes no

c) If you replied "yes", what result appears on the display?

for S = 10 .....

for S = 8 .....

for S = 25 .....

2. A school has decided to go to the theatre. The entrance ticket costs 5,000 lire each if the pupils total 50 or less than 50, and 3,000 lire each if they total more than 50.

   Write a programme which calculates and prints the total cost of the tickets according to the number of the pupils.

3. Write a programme which asks you to introduce two numbers A and B, and then if A is greater than B, it prints A - otherwise it prints B.

PRELIMINARY ANALYSIS OF THE PROPOSED TEST

The first exercise proposes a guided reading of the programme.

Point a) apparently not connected to the problem under consideration, aims at eliminating doubts on the difficulties concerning other instructions, which could be upstream and pollute the results (in the classes in question, the previous year, a research study was also made into the difficulties linked to the use of the instruction INPUT).

Point b) aims at seeing whether there are pupils who think "if S < 10" means that S can only be less than 10. In this case it can be concluded that the conceptual scheme was not mastered. It is believed that pupils who transform the programme into a substantially "mono-sequential" programme, reason in this way (type B of the previous paragraph).

Point c) presupposes the ability to put oneself in the computer's place and to follow the stages of its process in the various cases proposed, that is it asks for the capacity to decode the scheme at a semantic level. Any errors only on S = 10 can indicate that difficulty is limited to the logical negation of "less than", and are therefore not important for our analysis. It is also important to check the coherence between the answer to b) and to c). The lack of coherence should be examined case by case.

The exercises 2 and 3 propose the creation of simple programmes where a condition is present. In particular, 3 is extremely simple, and the pupil is asked to follow the scheme almost exclusively at operative level, while 2 implies the ability to apply the scheme in a very simple problem situation, and therefore to use it at conceptual level.

It is advisable to observe the relationship between 1 and 3 as well, since we consider that a correct answer to 3 and not to 1 can mean automatic repetition of the operative scheme.

RESULTS

Correct answers among 41 pupils of the third year class and 17 pupils of the second year class.

	3rd year class	2nd year class
exercise 1 a	28	12
b	25	6
c	13	4
exercise 2	12	6
exercise 3	14	6

Among the incorrect answers to exercise 1 point b, the choices were subdivided as follows:

"Only 8 can be introduced"	6 (out of 16)	6 (out of 11)
"Only 10 can be introduced"	1	0
"10 cannot be introduced"	4	1
"25 cannot be introduced"	2	2
"8 cannot be introduced"	2	2
"No value can be introduced"	1	0

The last pupil explained that he thought the programme was incorrect because of a syntax error he was not able to define.

For the answers to 1c, in the third year class, apart from the 13 who answered correctly, 2 other pupils can be considered as having correct answers in so much as there is an error which can be seen clearly as one of calculation (ex: 3*8 = 23); 2 other answers are coherent with 1b and mistaken only because the pupil thought he could not introduce the value 10, and other 2 are incorrect since the value 10 was multiplied by 3 instead of by 2.

Considering the same point for the second year class 4 pupils gave only incorrect answers for the value 10. As we stated previously, the errors concerning only the value 10 are classified according to the fact that we feel that the pupil does not clearly understand the difference between < and <=, or that he has some difficulty in negating < with >=.

So, concerning the correct reading of the instruction for programme selection, 19 answers can be considered correct in the third year class, and 8 in the second year class.

As far as problem 2 is concerned, 6 other answers in the third year class and 2 in the second year class can be considered correct from a conceptual point of view since they present errors of syntax, omission etc..

In the same way, other 11 results concerning exercise 3 in the third year class and 5 in the second year class

can be considered correct.

Therefore we have the following table (correct answers):

	3rd year (41 pupils)	2nd year (17 pupils)
1a	28	12
1b	25	6
1c	13 + 6	4 + 4
2	12 + 6	6 + 2
3	14 + 11	6 + 5

From a detailed study of the results it is evident that who gave a correct (or almost) answer to exercise 2, also replied correctly to all the points of question 1. The only exceptions are 2 pupils of the 2nd year class who gave an incorrect answer for 1b, not having understood the meaning of the question, as we found out during the following discussion.

This is not true for exercise 3, which was answered correctly by pupils who gave incorrect answers for 1c, which confirms the hypothesis of the preliminary analysis concerning possible repetition by heart of the operative scheme. We also noted that pupils who answered point 1b, saying that only value 8 could be introduced, did not answer question 2 or question 3, or answered in a way which was substantially "mono-sequential" (see preliminary analysis).

The results, moreover, are far better than those of previous year test (see paragraph 3).

In addition to the considered reasons, we maintain it is taken into consideration that this test was only on one subject and it divided the problem so that the pupil could learn while working.

During the discussion we noted that in all the classes the weaker pupils were not capable of supplying new interpretative keys for their errors. (The answers "I don't know", "I don't remember", "I made a mistake" etc. were frequent).

This is, in fact, the way the lack of scheme was expressed in the interview: the error cannot be declared because it does not exist as an error. At the most, the pupil can only realize that he has imitated the wrong model ("I thought I had to do it like the problem...").

CONCLUSIONS

After examining the protocols, and from the following discussion in class, we feel that in the group under consideration concerning the research problem - we can make the following classifications:

- pupils who "understood" the selection structure, both on the operative level (programming language) and on the conceptual level (they are able to transpose the scheme);
- pupils who, in spite of certain difficulties on the operative level, understood, in the main, the conceptual level;

- pupils who produce in an imitative way, but who do not master the conceptual scheme. These pupils are not able to decode satisfactorialy;
- there were no students who decode correctly but who cannot produce.

It would seem therefore, that the comprehension of the conceptual scheme which is the base for production, is necessary for decoding.

The problem seems to be essentially conceptual. The purely mastering of language, or, on the contrary, the difficulties associated with language could be far less important.

REFERENCES

Capponi B., Balacheff N., (1989), Tableur et calcul algebrique, *Educational Studies in Mathematics*, vol. 20, pp. 179-210.

Chevallard Y., (1985), *La transposition didactique*, La Pensée Sauvage, Grenoble.

Haussmann K., Reiss M., (1989), Aspects of declarative knowledge on control structures, *PME XIII Proceedings*, pp. 75-82.

Hoyles, Noss, (1987), Synthesing mathematical conceptions and their formalization through the construction of a Logo - based school mathematics curriculum, *International Journal of Mathematical Education in Science and Technology*, vol. 18, n. 4, pp. 581-597.

Pellerey M., (1986), *Informatica. Fondamenti culturali e tecnologici*, SEI, Torino.

Pellerey M. (Ed.), (1989), *L'informatica nella scuola media, come e perché*, SEI, Torino.

Reggiani M., (1988), A didactic proposal for the use of computers in the teaching of mathematics in the junior secondary school (age 11 - 14), *Proceedings of the first italian - german bilateral symposium on didactics of mathematics*, Pavia oct. 4 - 9, 1988, pp. 443-458.

Rogalski J., (1985), Alphabétisation informatique. Problèmes conceptuels et didactiques. *Bulletin APMEP* 347, pp. 61-74.

Rogalski J., (1987), Acquisition et Didactique des structures conditionnelles en programmation informatique, *Psychologie française* 32-4, pp. 275-279.

Rogalski J., Vergnaud G., (1987), Didactique de l'informatique et acquisition cognitives en programmation, *Psychologie française* 32-4, pp. 267-273.

Samurçay R., (1985), Signification et fonctionnement du concept de variable informatique chez des élèves debutants. *Ed. Studies in Mathem.* Vol. 16, n. 2, pp. 143-161.

Vergnaud G., (1985), Concepts et schèmes dans une theorie operatoire de la representation, *Psychologie française*, 30, pp. 3-4.

Vergnaud G., (1988), Theoretical frameworks and empirical facts in the psychology of mathematics education, *Proceedings of the sixth ICME*, Budapest, pp. 29-46.

Symbolising and Solving Algebra Word Problems: The Potential of a Spreadsheet Environment

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Whereas the research within 'traditional" algebra has highlighted pupils' difficulties with literal symbols it appears that pupils can more readily accept these as objects within a computer programming context. A spreadsheet provides a context for generalising from arithmetic and extending pupils' informal arithmetic strategies in ways which are not possible within their traditional "paper and pencil" work. This paper stems from a two year collaborative British/Mexican project which aims to investigate the ways in which pupils use a spreadsheet environment to symbolise and solve algebra word problems. Results of a pilot study carried out in Mexico with 8 pairs of pupils from the 6th grade of primary school and the 1st grade of secondary school indicate that pupils at a pre-algebra stage are able to develop skills to approach, in a numeric way, problems that conventionally are solved by algebraic tools. Unlike algebraic representation, a spreadsheet numeric approach requires subjects to keep analysing relations between unknowns and data, referring continually back to the problem context. The nature of this shifting between meanings derived from the symbolic system and meanings derived from the problem situation will be a focus of the main study to be reported on at the conference.

Introduction and Aims

For a number of years we have both in different and separate ways carried out research on the teaching and learning of algebra. Our differing research approaches have focused on history and epistemology with an emphasis on the evolution of symbolic algebra (Filloy & Rojano, 1984,1985,1989; Rojano & Colin, 1990) and on the potential of computer-based environments for learning algebra (Sutherland, 1988, 1989). We are now synthesising results from our previous work within a two year collaborative study which aims to:

- Investigate the way in which pupils use a spreadsheet environment to represent and solve algebra problems relating this to their previous arithmetical experiences and their evolving use of a symbolic language.
- Characterise pupils' problem solving processes along the dimension arithmetic/algebraic as they evolve through working in a spreadsheet environment.
- Develop and evaluate a didactical sequence to help pupils make links between spreadsheet and traditional algebra syntax.

Groups of pupils will be studied simultaneously both in Mexico and in Britain. These groups will consist of: two groups of eight pre-algebra pupils aged 10-12 years (one in Mexico and one in Britian): two groups of eight algebra-resistant pupils aged 13-15 years (one in Mexico and one in Britain). Within this paper we firstly present an overview of the theoretical background to the work and follow this with the results of a pilot study carried out in Mexico.

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We shall discuss the preliminary results of the main study during the conference.

Background

Much of the existing research on pupils' difficulties with school algebra has focused on their use and understanding of literal symbols. This research suggests that many pupils have difficulty in understanding that a letter can represent a range of values (Booth, 1984; Collis, 1974, Küchemann, 1981). They find it difficult to accept as an entity an "unclosed" expression in algebra (for example a+6) which relates to their difficulty in operating on these expressions (Booth 1984, Collis 1974). These misconceptions associated with the use of literal symbols present pupils with obstacles to the development of an algebraic approach to problem solving. Whereas the research within 'traditional" algebra has highlighted pupils' difficulties with literal symbols it appears that pupils can more readily accept these as objects within a computer programming context (Sutherland 1989, Tall 1989).

Algebra is the language of mathematics, a language which can be used to express ideas within mathematics itself, or within other disciplines. The development of algebra started at a time when even the unknown quantity was not represented by a symbol. Thus natural language was the only vehicle to express both the statement of the problem to be resolved and the resolution process. To solve problems was the main goal of the algebra of that time (from Babylonians to Diophantus). This goal changed as the symbolic algebra evolved to the point where algebra became the study of mathematical structures. But in school algebra the solving of problems remains an important objective. However, because mathematical problem solving and the manipulation of algebraic symbolism are usually introduced separately within the mathematics curriculum pupils are often unable to integrate their knowledge of these two domains. So for example they may be able to satisfactorily solve a symbolic algebraic equation but are unable to represent symbolically a mathematical problem. Much of the previous research on pupils' understandings of algebra has focused on helping pupils develop a better understanding of the mathematical problem, with the assumption that the algebraic symbolism will be a final added-on translation process. We suggest however that this separation of symbolism and referential meaning presents pupils with an obstacle to the effective use of algebra. We adopt an alternative position, namely that understanding of a mathematical problem develops as an inextricable negotiating process between symbols and referent.

There is substantial evidence that there is not a simple transition for pupils from an arithmetical to a more algebraic approach to problem solving. Filloy and Rojano point to the "existence of a didactic cut along the child's evolutionary line of thought from arithmetic to algebra. This cut corresponds to the major changes that took place in the history of symbolic algebra in connection with the conception of the "unknown" and the possibility of "operating on the unknown" (Filloy & Rojano, 1989, P 20). When we turn to the computer programming context we find that although a similar barrier to using a symbolic language to express generality exists (Sutherland, 1989), this

barrier can be more readily overcome in this environment than in a paper and pencil environment.

There are two main reasons why the influence of the computer could radically change the ways in which pupils learn algebra. The first is that computer-based experiences can provide pupils with a background of using and manipulating symbols offering the potential of a new entry point into algebra. The second is that mathematics itself is being influenced by the computer as in for example the development with fractals and non-linear feedback systems (Stewart, 1989). To interact with most computer environments it is necessary to express generality in a symbolic language and this is particularly the case for programming languages of which a spreadsheet package is an example. There are a number of reports of pupils using these formal computer-based environments in ways which are far more sophisticated that that which is normally expected of them within school algebra (see Healy et al 1990). This is not to suggest that pupils use of these environments is unproblematic but only that with appropriate support pupils can use them in ways which far exceed their use of school algebra. Whether this is because these environments are qualitatively different from "paper and pencil" algebra or whether it is because expectations of pupils' performance within school algebra is restricted are questions which will be addressed by this research study.

Spreadsheets have been found to be valuable from the point of view of developing algebraic understandings (Capponi & Balacheff (1989), Healy & Sutherland, 1989). Unlike other programming languages naming and declaring a variable is no longer a focus in a spreadsheet environment and a mathematical relationship can be encapsulated by physically moving the mouse or the arrow keys without explicit reference to the symbolic spreadsheet language. In this way the environment allows pupils to represent and test out mathematical relationships without having to take on board all the complexities of a symbolic language, although they can see this relationship represented within the spreadsheet environment itself. The algebraic relationships are likely to be closely related to arithmetic and in this sense a spreadsheet provides a context for generalising from arithmetic and extending pupils' informal arithmetic strategies in ways which are not possible within their traditional "paper and pencil" work.

When considering the potential of the computer for the development of algebraic thinking it is important to investigate both pupils' conceptions within the computing context and the links they make to a traditional school algebra context. Most previous computer-based studies have been weak in this respect although Sutherland (1989) found that the links pupils make depend very much on the nature and extent of their computer based experiences. This emphasises the need for studies to be carried out to find good problem situations in algebra which are both challenging and motivating for pupils who have already had experience of using a computer-based formal language.

Results of the Pilot Study

We report here the results of a preliminary study carried out in Mexico with eight pairs of children (four from the 6th grade of primary school and four

from the 1st grade of secondary school) who were introduced to the spreadsheet package LOTUS 1-2-3 as a means of solving algebra word problems (see Figs. 1-4.). Before carrying out the spreadsheet tasks the pupils had not received any instruction on solving these kinds of algebra word problems. The pupils were chosen after examining the results of a pre-questionnaire which was administered to 123 students. Three aspects were checked through the questionnaire a) interpretation of a word problem statement; b) the concept of lowest common multiple; and c) the notion of functional variation, specifically proportional variation. By analysing the questionnaire results three levels of proficiency were identified and the pupils were chosen from different levels. Pupils from the higher level could correctly interpret word problem statements and in each case proposed a reasonable method of solution. Concerning the notion of functional variation they could generate specific output values of a proportion function from input values and could also generate specific input values given the output values. Pupils in the medium level could interpret correctly some of the word problem statements but could not give a strategy for solving them. In the functional variation items they could only generate output values given the input values. Pupils in the lower level could only interpret the simplest word problem statements and in the functional variation problems they incorrectly constructed output values from the given input. The notion of Lowest Common Multiple was absent in all pupils of the study. Eight pairs of pupils were chosen across a range of attainment as measured by the questionnaire. Each pair was from the same school grade and the same proficiency level.

The teaching experiment consisted of 7 two-hour sessions, the first of which was devoted to teaching pupils how to generate a column of numerical data (introducing a general rule, for instance, to generate the first 20 multiples of 2) and how to locate specific data within the rows and columns of the spreadsheet. In the second session a group of lowest common multiple problems (see for example the LCM problem in Fig. 1) were used with the idea of teaching pupils to locate the answer to a problem, by means of exploring data in several columns simultaneously. In the third session, proportion word problems were presented to children as a basis for the introduction of work with equations. In this session problems which could be represented by one step equations were used (see for the one step equation problem in Fig. 2). In the last four sessions two blocks of algebra word problems were used, with the aim of observing to what extent these pupils, at a pre-algebra stage, were able to approach, in a numerical way, problems that conventionally are solved algebraically, by means of a system of 2x2 or 3x3 linear equations (see for example the algebra word problems in Fig. 3).

Fig. 1 *Example of Lowest Common Multiple Problem*

John's motor cycle uses one litre of gasoline per 16 kilometres and Sandra's uses one litre per 24 kilometres. What is the minimum number of litres which they need to drive the same distance.

Fig. 2 *Example of One Step Equation problem*

The width and length of a rectangular piece of land are 6m and 27m respectively. What are the dimensions of a piece of land with an equivalent area, but having its width equal to its length?

Fig. 3 *Example of Algebra Word Problems:*

Problem T :For a theatre play, children's tickets were sold at $80 pesos each and adults' tickets at $120 pesos each. The number of children's tickets sold was 100 more than the number of adult tickets. The total amount of money collected for the show was $30,000 pesos. How many tickets of each sort were sold?

Problem OB : 91 objects are going to be distributed among 3 people, in such a way that the first of them will receive 3 times the number given to the second one, and the second one will receive 3 times the number given to the third one. How many objects will each person receive?

Problem CF: A merchant mixes 40 kilograms of coffee which is sold at $400 per Kg. with a certain amount of another sort of coffee which is sold at $600 per Kg. How many Kilograms of $600 coffee should he employ to obtain a mixture of coffee to be sold at $500 per Kg.

Lowest common multiple problems (LCM). In all cases problems were presented in realistic situations, litres of gas/distance; speed/distance; intermittent lights; gears (see for example Fig. 1). Evidence from the questionnaire suggested that the pupils did not understand the notion of a lowest common multiple at the beginning of the experiment. The main results from this study were that:

- Pupils with a high proficiency level were able to transfer the strategy taught for the first problem (LCM of two numbers) to more complex cases (i.e. finding the LCM of 3, 4 and 5 numbers). This involved exploring several numerical columns simultaneously and generating data beyond the first screen.

- Pupils with a medium proficiency level needed support from the teacher in order to make the transference described above. These pupils did not relate their answer to the problem context and worked only within the spreadsheet environment. Once these pupils could express the problem statement in numeric figures within the spreadsheet environment, they did not need to recover the original meaning of the numeric figures they were dealing with. This could imply that these pupils were developing and using an abstract version of the lowest common multiple notion, once the problem had been expressed in the numeric domain.

- Pupils with a low proficiency level kept displaying difficulties at a technical level (e.g. with copying a rule) indicating that they had not had enough time to learn how to manipulate the spreadsheet environment within a problem situation with which they were confident. They tended to automate the way they dealt with the data and tried to do exactly as they had done in the first problem presented.

level could use the spreadsheet to solve these problems. One method of solving Problem T using a spreadsheet is presented in Fig. 4.

When solving Problem T the pupils needed support in generating the first two columns, i.e. those representing the adults and children's tickets (see Fig. 4). Then the students themselves generated the corresponding columns for the money collected in each case using the rules: A4*120, A5*120 etc; and B4*80,B5*80 etc. Pupils felt the necessity of reading again and again the problem statement in order to recover the meaning and sense of summing up all the partial columns. Some help was required to generate the column of total amounts of money (addition of the last two columns), which was then explored by pupils to find out the problem's solution. Later on, when the students were presented with similar problems they could easily apply the former approach to solving them. When however the complexity was increased, in terms of the number of equations and unknowns of the system underlying the problem, as well as the kind of relationships between unknowns and data, pupils had to return to an analysis level in order to understand the variants introduced in the new situation. Once they achieved a spreadsheet representation of the partial relations present in the problem, the main difficulty was to conceive one last complete relationship which involved relations on and between the elementary ones previously built up, that is, the equivalent of a "complete" equation

At this stage it can be said that children are able to generalise the use of a spreadsheet to problems involving 3 or more equations and unknowns, provided that they can build up a key relationship which permits location of the totality of solutions. This implied, in most of the cases in this study, a mental (or paper and pencil) analysis on the part of the students. Unlike algebraic representation, a spreadsheet numeric approach requires subjects to keep analysing relations between unknowns and data, referring continually back to the problem context, at least for families of problems like Problem OB. Solving this problem in the spreadsheet environment involved:

1st step- generating the natural numbers in column A
2nd step -generating column B with a rule of the form (B1 = A1 *3)
3rd step - generating column C with a rule of the form (C1 = B1*3)
4th step - generating column D with a rule of the form
(D1 = A1 + B1 + C1)

Pupils needed to go back to the semantics of the problem between the 3rd and 4th step. This would not necessarily be the case if the problem was solved using algebraic representation. The nature of this shifting between meanings derived from the symbolic system and meanings derived from the word problem itself will be a central focus of our main study.

Concluding Remarks

Pupils at a pre-algebra stage are able to develop skills to approach, in a numeric way, problems that conventionally are solved by algebraic tools. In particular the spreadsheet environment is valuable for solving algebra word problems which involve decimal solutions.

Two turning points were detected in the pupils' evolution path.: one of them is when pupils have to transfer their solution method to problems

involving more than 3 unknowns and relationships — this can be a reference point with regards to the relevance of algebra as a synthetic representation of a problem which can be solved at a pure syntactic level. The second turning point is when neither a spreadsheet nor algebra is sufficient to solve a problem situation without an analysis of the elements of the statement through the entire resolution process — this is the case in the coffee problem (problem CF).

As far as learning to use the spreadsheet is concerned we observed that pupils need firstly to become familiar with manipulating a spreadsheet environment (i.e. entering and replicating rules) before they can use the spreadsheet to express and solve algebra word problems. This takes time for some pupils. Pupils also need to know how to make sense of the results obtained within the spreadsheet environment with reference to the problem statement and without turning to the teacher for the correct answer. This means that the role of the teacher is critical in the introductory stages of spreadsheet work.

References:

BOOTH, L., (1984a), *Algebra: Children's Strategies and Errors*, NFER-Nelson, Windsor.
COLLIS, K.F., (1974), Cognitive Development and Mathematics Learning, paper prepared for *Psychology of Mathematics Education*
CAPPONI, B. & BALACHEFF,N. (1987) "Tableur et Calcul Algebrique", *Educational Studies*, Vol. 20, No. 2.
FIILLOY, E. & ROJANO, T. (1984), :La Aparicion del Lenguaje Aritmetico Algebraico. *L'Educazione matematica, Anno V*, Vol. 3, Dec., pp278-306
FILLOY, E. & ROJANO, T. (1985) From Arithmetical to Algebraic Thought (A Clinical Study with 12-13 year olds) *PME-NA*, Wisconsin, pp51-56.
FILLOY, E. & ROJANO, T. (1989) Solving Equations: The Transition from Arithmetic to Algebra. *For the Learning of Mathematics*, 9, 2, June.
HEALY L , HOYLES C.& SUTHERLAND R.(1990), Peer Group Discussion in Mathematical Environment: *Final Report to the Leverhulme Trust*, Institute of Education University of London
HEALY, S., & SUTHERLAND, R., 1990, *Exploring Mathematics with Spreadsheets*, Blackwell.
KÜCHEMANN, D., (1981), 'Algebra' in HART, K., (Ed), *Children's understanding of Mathematics: 11-16*, John Murray: London.
ROJANO,T. & COLIN J., (1990) Bombelli, La Sincopacion del Algebra y la resolution de Ecuaciones. *Enviado a l'Educazione matematica*. Dec, 1900
SREWART, I (1989) *Does God Play Dice? The Mathematics of Chaos*, Penguin Books.
SUTHERLAND, R (1988), *A Longitudinal Study of the Development of Pupils' Algebraic Thinking in a Logo Environment*, Unpublished Doctoral Thesis, University of London, Institute of Education
SUTHERLAND, R., 1989a, *'The Gap between Arithmetical and Algebraic Thinking: computer effects'*. Proposal to the Economic and Social Research Council Project, Department of Mathematics, Statistics & Computing, Institute of Education, University of London.
SUTHERLAND, R., 1989b, Providing a Computer-based Framework for Algebraic Thinking, *Educational Studies in Maths*, Vol. 20, no. 3, pp 317-344.
TALL, D., 1989, Different Cognitive Obstacles in a Technological Paradigm, in Wagner, S. & Kieran, C., *Research Issues in the Learning and Teaching of Algebra*, Hillsdale: LEA.

DAMIEN: A CASE STUDY OF A REORGANIZATION OF HIS NUMBER SEQUENCE TO GENERATE FRACTIONAL SCHEMES

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Damien is a third grader that participated in 17 teaching episodes during a teaching experiment conducted in an elementary school in Georgia. This case study provides an interpretation of Damien's ways of operating to generate fraction schemes as a result of the modification of his operations and strategies with natural numbers.

This article provides an interpretation of the behavior of a third grader when solving problems with natural numbers and using his number sequence to solve fractional tasks. The theoretical basis of this case study is that children actively generate knowledge by reorganizing prior cognitive structures (Piaget, 1985). Damien's behavior, answers, and explanations indicate that he used his number sequence to generate partitioning schemes.

Theoretical Rationale

Implicitly or explicitly defined, different ways of partitioning were the integral features of the problems posed to Damien to investigate his ways of operating. Such problems required a partition of the whole to generate a unit fraction and the multiplicity of the unit fraction to recast the whole. McLelland and Dewey (1895) point at the dual and intrinsic need of the mind for dividing and multiplying to generate fractions: "Fractions employ more explicitly both the conceptions involved in multiplication and division-namely analysis of the whole into exact units, and synthesis of these into a defined whole" (pp.241-242). That is, some of the mental operations used in

the conceptualization of natural numbers are the operations that have to be extended to generate fractional numbers. It is by no coincidence that Courant and Robbins (1969) assert that "the generalization from the natural to the rational numbers satisfies both the theoretical need for removing the restrictions on subtraction and division, and the practical need for numbers to express the results of measurement" (p. 56).

Steffe and Cobb (1988) have characterized different types of mental operations that are the result of different levels of children's abstractions in the process of constructing their number sequences. It seems that the most advanced number sequence that the children construct is the explicitly nested number sequence. In this sequence the child can see the relation of inclusion symbolized in the sequence and can intentionally disembed a segment of it without destroying the records of counting in the original sequence. As a result of mental operations of different degrees of sophistication, the child creates different types of units. Among these units the most refined units seem to be the composite units and the iterable units. A child generates a composite unit when he is able to focus on the unit structure of a pattern leaving implicit the oneness of its constituent elements. A child creates an iterable unit when he uses "the composite unit to organize a collection of items into an indefinite number of composite units, without making the composite units at the sensory-motor level" (Steffe, 1990, p.18).

Methodology

The teaching experiment is a methodology designed to formulate explanations and models of children's mental constructions based on intensive verbal interactions between the researcher and each student. The models formulated are based on inferences drawn on observations of children's constructions of mathematical objects. These models are based not only on the child's behavior, but also on the researcher's conceptual elements that play a role in the interpretations of the child's ways of operating (Steffe, von Glasersfeld, Richard, and Cobb, 1983, p. xvii).

From the seventeen teaching episodes (60 minutes each) in which Damien participated during five and a half months, three were to investigate his ways of operating with natural numbers, one was to explore his initial concepts about fractions, and the remaining thirteen were to help him to develop his initial knowledge of fractions.

Damien's Knowledge of Number Sequences

Damien operated with composite and iterable units at the level of the explicitly nested number sequence (Steffe and Cobb, 1988). Damien counted by twos, threes, fours, fives, and tens using each unit as an iterable unit. The following dialogue shows that he displayed a double counting activity which indicates that he was operating with abstract composite units and treating them as entities to be counted.

T: Would you please count by fives starting at twelve and up to thirty-seven? D: 12-17-22-27-32-37. T: How many fives did you count? D: Five. T: How did you find your answer? D: Twelve. Twelve and five is seventeen. That's one. Seventeen and five is

twenty-two. That's two. Twenty-two and five is twenty-seven. That's three. Twenty-seven and five is thirty-two. That's four. Thirty-two and five is thirty-seven. That's five.

Damien segmented his number sequence using his concept of five while keeping track of the number of segments being made by double counting. Moreover, he united each segment with the preceding segment and used his uniting operation progressively.

In solving a division task, Damien used his progressive integration operation and composite units as the following dialogue shows.

T: Suppose that there are twenty-four students in your class and each group of four students eats one pizza. How many pizzas are needed? D: [After some time] Five. T: Why? D: Four is one; eight is two; twelve is three; sixteen is four; twenty is five; twenty-four is six. Six.

His answer indicates that, prior to iterating, he viewed twenty-four as a composite unit segmented in composite units of four that he could iterate. That is, he could increment four more ones by incrementing one four while keeping track of the number of iterations by double counting.

Damien's Initial Knowledge About Fractions

Damien's solutions to the following and analogous tasks indicate that he has yet to increase his awareness of the like size of the parts as the necessary condition to have a multiplicative part-whole relationship.

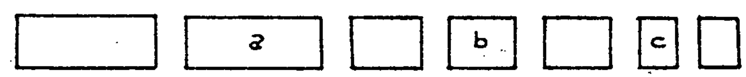

Figure 1

When Damien was shown the aligned pieces in Figure 1 and referred to them as a train with different size cars, he counted

all the pieces and said that either <u>a</u>, <u>b</u>, or <u>c</u> was one seventh of the train. However, after he was made aware of the different size of the pieces, he overlapped the smaller pieces over the larger pieces and found that the piece <u>c</u> was "one sixteenth," the piece <u>a</u> was "four sixteenths," the piece <u>b</u> was "two sixteenths," and that the two larger pieces together were "eight sixteenths."

Damien's answers to this and other similar tasks indicate the he had the construction of what I call **iterable part-whole scheme** underway since he reconstituted the whole sometimes as a plurality of parts of different size and sometimes as a plurality of parts of the same size based on which the part acquired a "fractional" quantification.

In general, Damien indicated a strong awareness between the numerosity of the parts in the whole and the quantitative description of the part according to such numerosity. However, he was less aware of the necessity of the equal size of the parts in order to have a multiplicative part-whole relation that generates a unique fractional quantification of the part. In this sense, Damien had yet to modify his scheme.

Damien's Multiple-Units Coordinating Scheme

Damien independently used the content of the fractional parts to compare them. He exchanged a one-hundred-dollar bill into fifty, twenty, ten, five, and one-dollar bills and put the bills on the table.

T: Which is bigger, two tenths or one fifth? D: [Looking at the bills] They are the same. T: Why? D: Because two tenths is twenty, and one fifth is twenty. T: Which is bigger, ten twentieths or five tenths? D: Ten twentieths. [Pause] They are the same. T: Why? D: Because ten fives is fifty, and five tens equals fifty.

By his own initiative, he used the content of the fractional parts of one hundred dollars as a means to compare them. Referring to the numerosity of the fractional parts indicates that his comparisons were not linked to the inverse relation between the size of each part and the number of parts nor to the coordination between partitions. This indicates that his answers were the result of coordinating fractional and whole number units.

In the following dialogue, Damien shows an awareness of the difference between the numerosity within the part and the numerosity of fractional parts in the unity or whole (one hundred).

T: Twenty dollars is one tenth of the amount of money that I have hidden. What is that amount of money? D: [Taking some time to think] Two hundred dollars. T: Why? D: Because five twenties is one hundred, and five more is one hundred. Five plus five is ten, and one hundred plus one hundred is two hundred.

Damien's explanation "five twenties is one hundred, and five more is one hundred. Five plus five is ten, and one hundred plus one hundred is two hundred" indicates his awareness of the number of parts (ten) and the numerosity of each of the ten parts (twenty) to reconstitute the whole or unity.

Damien was also able to generate equivalent fractional parts on his own. Within the context of a hundred pennies that he divided into 2, 4, 5, 10, and 20 cups, the following dialogue took place.

T: Suppose that with one dollar you buy this car for sixty cents, and this ball for fifteen cents. What part of your money would you use? D: [Taking some time] Three fourths. [Pause] Oh! Fifteen twentieths [then he counts by fives up to seventy-five while passing cups from the group of twenty cups, like verifying his answer]. Seven tenths and one twentieth. Ten is one tenth,

> *twenty is two tenths, ..., sixty is six tenths, seventy is seven tenths, and five is one twentieth.*

The above dialogue highlights Damien's ability to generate equivalent fractional parts as the result of using two different partitions of the same whole or unity (one hundred pennies) by means of different segmentations (fives and tens) of the number sequence from one up to one hundred. He generated a double counting strategy in which he, at the same time, kept track of the numerosity of unit fractions, and the numerosity that symbolized the actual content of each fraction. This indicates his awareness of the different types of units (whole number units and fractional units) he was working with and the different partitions in which the whole could be segmented.

In the context of money, Damien seems to have constructed a **multiple-units coordinating scheme** because, given an amount of money, he used his number sequence to segment this quantity in two or more ways and was able to discriminate between the number of segments, the inner numerosity of the segment, and the fractional quantification. That is, he was able to view whole number units as fractional units in relation to a numerical whole. However, this scheme had yet to develop in contexts of collections of objects and other discrete contexts for Damien to be able to abstract the meaning of equivalent fractional parts.

Conclusion

The first interview about fractions showed that, when the teaching experiment started, Damien could reconstitute the whole as a plurality of parts. However, he had no conscious awareness of the like size of the parts as the necessary condition to

reconstitute the whole as a multiple of a part. Nevertheless, as a result of this numerosity, he was able to quantify a part as a "fraction" of the whole or unity.

During the course of the teaching experiment, he developed an awareness of the equality of the size of the parts to recast the whole as a multiple of the part. By working with different partitions, Damien became aware of the inverse relation between the size of the part and the number of parts in each partition Comparisons of fractions and generations of equivalent fractional parts were done by means of coordinating fractional and whole number units. In general, it can be said that Damien used his explicitly nested number sequence and modified his operations and strategies with the natural numbers to generate his fraction schemes.

References

McLelland, J. & Dewey, J. (1895) The psychology of number New York: D. Appleton & Co.

Piaget, J. (1985) The equilibrium of cognitive structures: The central problem of intellectual development. Chicago: The University of Chicago Press.

Steffe L. P., von Glasersfeld, E., Richard, J., & Cobb, P (1983) Children's counting types: Philosophy, theory, and application New York: Praeger.

Steffe, L. P. & Cobb, P. (1988) Contruction of arithmetical meanings and strategies. New York: Springer-Verlag

Steffe, L. P. (1990) Construction of multiplying of dividing schemes. (Manuscript copy)

THE USE OF LANGUAGE IN THE CONTEXT OF SCHOOL MATHEMATICS

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ABSTRACT: The language used in mathematics has a number of features particular to it that are not familiar to pupils and this can be a source of confusion. This paper attempts to point out these features and suggests the term "sub-language of mathematics" as appropriate to refer to this types of language, thus drawing attention to the influence of the subject matter on the language used in its context.

1. Background

This paper sets out to make the case that it is appropriate to consider that the language used to surround the communication of mathematical ideas, has characteristics proper to itself and thus can be distinguished from everyday language.

In school, and in the context of a particular subject matter, language often appears to present differences in many ways from the language the child encounters outside of the class and the school. Richards (1978) analysed lessons of various subjects at the primary and secondary levels and concluded that children in school encounter language forms which vary considerably from those of the everyday language. These forms are more formal, they use specialised vocabulary (particularly, for example, in science and mathematics) and contain certain types of repetitive patterns of a syntactic nature, sometimes more complex than those which the child normally uses.

Mathematics is involved in many everyday activities and pupils acquire and abstract many of the notions of the subject through these or planned school experiences. However, the development of these notions to more advanced levels does not proceed automatically nor by a more organised approach. Skemp (1986) argues that mathematics cannot be learned directly from the everyday environment but only through others, like teachers, and through the child's own "reflective intelligence". This suggests that communication and particularly language plays a role in the context of the school mathematics.

Adda (1986) believes that children do not normally come into direct contact with mathematics but become familiar with its ideas mainly through the intermediary of teaching. But the teaching of mathematics and thus its understanding is carried out mainly through the use of the natural language and this attributes particular importance to the language medium. Clark

(1974), considering the role of language in mathematical learning, discovery and communication, recognises that children construct mathematical concepts originally through concrete experience and language can help in that through discussion and instruction in the classroom.

The above views underline the fact that although language is not all that matters in mathematics as a school subject, it certainly is an essential input in pupils' understanding of the subject matter.

2. The use of the concept of language in the research literature

The terms "language of mathematics" and "mathematical language" have often been used in thinking about the interaction between language and mathematics. However, this terminology can give rise to many problems because:

(i) it does not specify the content, that is, whether it refers to the natural language used to talk and write about the subject, or the symbolic system of mathematics or both and is sometimes used to refer to all or any one of these three areas;

(ii) it does not specify the context, that is, whether it refers to pupils, teachers, or professional mathematicians. Austin and Howson (1979), talking about the interaction between language and mathematics, express the same concern and propose the following distinction: a) language of the learner, b) the language of the teacher/author and c) the language of mathematics. In each of these cases the English language plays the role of tool for interaction which is adapted for use with different strategies to deal with each of the above cases;

(iii) "language of mathematics" incorporates an assumption that such a "language" possesses the characteristics or principles of a language. This suggests that it may be treated as a language, thus over-emphasising common features which are shared with a language and underestimating the uncommon ones.

Love and Tahta (1977) advocate that the teaching and learning of mathematics involves communication and this inevitably invokes ways of thinking about language. For them, the term "language" in the expression "language of mathematics" is referring to the ordinary language which by definition is part of the curriculum for any subject They argue that mathematics is not itself a language but is a particular use of parts of the everyday language. The language used in the context of mathematics is really a "language within a language' (Pimm, 1987).

The term "language" is sometimes used to refer to the mathematical symbolism. Much scepticism has been expressed about the idea of seeing the formalisation of mathematical ideas as a language on its own. Sinclair (1980) recognises that although there are similarities between the symbolic system of mathematics (arithmetic) and natural language, differences are striking. For example, natural speech is highly context-bound, redundant and takes for granted a number of assumptions, whereas mathematics is non-redundant, mostly explicit and unambiguous. In addition, there is a lack of a one-to-one correspondence between how one writes or says things in natural language and how one writes them in terms of the symbolic system of mathematics. Austin and Howson (1979) refuse to accept that mathematics (i.e. symbolic system) is a language in the sense of being a "communication tool" because the knowledge involved can be expressed in a variety of "languages" from an everyday language to the language used by Russell in "Principia Mathematica".

Summarising, the term "language" has been used in a variety of ways with reference to mathematics and particularly to school mathematics, although this usage is not always clear. Thus, Wheeler (1983), understandably states in an often quoted remark that " I shall keep away from the region signposted *Mathematics is a language*. I believe it to be uninhabited".

3. Features of the language used in school mathematics

From what has been said up to now, it appears that the everyday language as it is used in mathematical discourse has some particular features and functions. Kane (1970), discussing the particularity of the language used in mathematics contexts, points out a number of differences between mathematical English (ME) and ordinary English (OE):(a) ME is on the whole non-redundant, whereas ordinary language is by necessity redundant (b) there are words and phrases with different meanings in ME and OE; (c) grammar and syntax are not as flexible in ME as in OE. In other words, the interaction of language with mathematics seems to attach a number of special features to the English used to express mathematics which either do not exist at all in the ordinary English - or they are not of importance for the language in its everyday use. Below, a first attempt is made to summarise some of these features at three levels: semantic, lexical and syntactic.

3.1. Semantic features

(a). **Extension of grammatical functions:** A number of words when used in the mathematics context acquire additional grammatical functions, for example, the nouns "sum" and "times" of everyday use become transmuted into verbs in the mathematical context.

(b) **Nature of objects:** Nouns used in the mathematical context do not usually designate human beings, in other words, the "human" feature is absent. However some verbs can be thought of as taking human subjects as implied by the use of imperatives, for example,"factorise", or "solve". This absence of a human characteristic in language is also marked by the absence of the personal pronoun in the mathematical discourse.

(c). **Specificity/generalisation: quantifiers:** The importance and frequent use of quantifiers is self-evident. As Freudenthal (1973) notes:"mathematicians discovered the almost self-evident fact that logic cannot do (deal) with the subject-predicate-structure, but badly needs relation-and quantifier-structure". For example, "the letter x stands for any number". Also in mathematics, quantifiers such as "some", "all", "any", "every", "there is" etc, have a more precise meaning. Although everyday language also makes frequent use of quantifiers, they are not so abstract since they are cloaked by expressions such as "always","everywhere"

(d).**The names of numbers - nominal and adjectival uses:** In mathematics, number names can have two functions: either adjectives as in "three apples" or "three variables", or nouns as in "the number *three*" or "the value of the unknown is *three*". In everyday discourse number names nearly always function as adjectives (Pimm, 1987, Munro, 1977) and rarely as nouns.

(e). **Imperative:** The extensive use of the imperative form in the mathematical discourse - the "mathematical imperative" - is a well known phenomenon (Pimm, 1987) e.g. "remove", "consider" etc. This use occurs in ordinary discourse too, but only in certain contexts, designating particular roles and functions. The imperatives used in the context of mathematics can be divided into two categories: assumption and instruction imperatives. The former "establishes the hypothetical nature of a sentence in an argument" (Nederpelt , 1982). For example "suppose a=1"....The latter - instructional imperatives - can have a certain mathematical function, e.g. "remove the brackets", "solve", "work out" etc, In addition, a number of these imperatives are used to refer only to certain parts of a mathematical object, in contrast with those which concern objects as wholes. For example, the verbs "solve" and "simplify" refer to an equation or to an algebraic expression respectively as a whole, whereas the verbs "remove" or "substitute" concern only certain parts of the equation or expression All these verbs can be found in everyday vocabulary, but in the context of mathematics, they designate a certain mathematical activity.

- (f).**Tenses of verbs:** Miller and Lenneberg (1978), point to a convention which appears when talking "about mathematical objects and their properties", namely the use of the present tense as the dominant tense. Other tenses are mainly used when everyday problems are discussed as in the application of concepts and ideas.

(i).Mathematical words. These are technical words with a special mathematical meaning, found primarily in the context of mathematics, e.g. "equation", "add", "exponent", etc. (Munro, 1977, Halliday, 1978, Pimm, 1987).

(ii).Words common to both the everyday and mathematical language: These words can be separated into two categories: (a) Words which have different meanings in mathematics and everyday language: These are ordinary language words which, when used in the mathematical context, are given a new (mathematical) meaning. For example, "difference", "remove the brackets", etc. (b) Words which are used in both the everyday and the mathematical context but with a slightly refined meaning in the latter. That is, everyday language words which acquire a mathematical flavouring, thus slightly altering their original meaning e.g., "substitute" (Pimm, 1987).

(iii). Absence of certain kind of words: There are a number of categories of words which do not appear in the mathematical discourse. For example, there are no words (verbs or nouns) expressing feelings, beliefs, etc, such as "hope", and "feel".

3.3. Syntactic features

A number of researchers have pointed to the fact that the language structures peculiar to mathematics are different from those of the everyday language and pupils have to get accustomed to these. Some of these features are:

(i) The written language used in mathematics context does not make extended use of questions, particularly tag questions, e.g."you should never divide by zero, should you?".

(ii) Exclamatory sentences are not used.

(iii) there is frequent usage of modals, for example, " you SHOULD remember that.." and "x CAN be replaced by...". The use of modals reflects the need to conform to mathematical rules and the use of the notion of possibility, activities commonly used in the mathematical discourse.

4. The concept of "sub-language"

Any kind of communication relevant to mathematics involves the English language but in a "mode" which is distinct from other types of linguistic communication and often entailing a considerable exposure to symbols and symbolic manipulations. But what are the particular features that make the language used in the context of mathematics distinct from the language used in other contexts?

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Recently, researchers in linguistics, recognised the fact that there is some relationship between a subject and the way it is expressed in a language. Various investigations into scientific articles, technical manuals, legal documents and other written material have suggested that in each case, English is used systematically , with a specialised vocabulary. This led to the notion of a "sub-language" within a natural language. The term appears for first time in Harris's (1968) "Mathematical Structures of Language" where he states that "certain proper subsets of the sentences of a language may be closed under some or all of the operations defined in the language, and thus constitute a sub-language of it". This definition was later revised to also include shared habits of word usage on the part of the speakers. In other words, a new term or a sentence with a certain grammatical construction is considered to be part of a sub-language if its use has been "conventionalised by the community of specialists" that use it. Kittredge and Lehrberger (1982) note that the term "sub-language" is currently used to refer to "a set of sentences whose lexical and grammatical restrictions are a reflection of restricted sets of objects or concepts and their relations encountered in a particular domain or field". Lehrberger (1982) puts forward the following characteristics for what he calls a "sub-language":(i) limited subject matter; (ii) lexical, syntactic and semantic restrictions; (iii) "deviant" rules of grammar (i.e sentences which are considered grammatically illegal in the standard language, or rules which describe co-occurence restrictions which do not not exist in the standard language); (iv) high frequency of certain constructions; (v) text structure which uses certain characterisic pointers (vi) use of special symbols.

The characteristics of a "language of mathematics" as presented in section 3 would seem to indicate that mathematics may be considered a sub-language in Lehrberger's sense.

4. Conclusions

The language used in the context of mathematics is quite distinct from the language used in other contexts. This is due to a certain degree to its abstract nature and also to a number of linguistic practices established over the years. Research seems to indicate that some pupils' problems with the subject matter can have their source in the peculiarities which characterise this language and which can give rise to linguistic demands which are different from those which the child encounters in everyday life. A new term is needed which captures these peculiarities thus allowing the detailed investigation of their linguistic demands on pupils' and at the same time is free from false assumptions that the label 'language" might provide. The term "sub-language of mathematics" appears to serve reasonably well this purpose.

References

Adda, J. (1986), " Fight Against Academic Failure in Mathematics", in P.Damerow et al (Eds), *"Mathematics for All"*, Report and papers presented in Theme Group 1 of the 5th international Congress on Mathematical Education, Adelaide, 1984.

Austin , J.L.and Howson, A. G. (1979), "Language and mathematical Education", *Educational Studies in Mathematics*, 10, 161-197.Clark, R. (1974), "Some aspects of psycholinguistics", Final Report of the Symposium *Interactions between linguistics and mathematical Education,, Nairobi, Kenya*, Copenhagen.

Freudenthal, H. (1973), Mathematics as an Educational task, Dordrecht, Reidel.

Halliday, M.A.K. (1978), *Language as a social semiotic: The social interpretation of language and meaning*, London, Edward Arnold.

Harris, Z (1968) *Mathematical Structures of Language*, New York: John Wiley & Sons

Kane, R. B (1970) The readability of mathematics text books revisited.*The Mathematics Teacher*, Vol 63, no.7, p.579-581

Lehrberger, J., "Automatic translation and the concept of sublanguage", in R. Kittredge and J. Lehrberger, *Sublanguage*, Berlin, N. York, Walter de Gruyter.

Love, E. and Tahta, D. (1977), "Language across the Curriculum: Mathematics", *Mathematics Teaching*, 79, 48-49.

Munroe J (1977) Language abilities and mathematics performance *Australian Journal of Remedial Education* Vol.9 no.3 p.24-31.

Nederpelt, R.P. (1982), The signal value of words in mathematical phrasing, *Proceedings of the 6th Conference for the Psychology of the Mathematical Education*, Antwerpen, 108-111.

Pimm, D. (1987), *Speaking mathematically: communication in Mathematics Classrooms*, London, N. York, Routledge and Kegan Paul.

Sinclair, H. (1980), Young children's acquisition of language and understanding of mathematics, *Proceedings of the fourth International Congress on Mathematical Education*, edited by M. Zweng, T. Green, J. Kilpatrick, h. Pollak and M. Suydam, Boston, Birkhauser.

Skemp, R. (1986), *The psychology of learning mathematics*, 2nd edition, Penguin.

Richards, J. (1978), *Classroom language: what sort ?*, G. Allen and Unwin.

Wheeler, D. (1983), Mathematics and language, *Proceedings of the Canadian Mathematics Education Study Group*, annual Meeting, Charles, Verhille (Ed.), 86-89.

EMERGENT GOALS IN EVERYDAY PRACTICES:

STUDIES IN CHILDREN'S MATHEMATICS

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This paper presents a research model for the study of the interplay between culture and the development of mathematical understandings in children. The model consists of 3 components: (1) analyses of goals that emerge in cultural practices, (2) analyses of the of form-function shifts linked to children's efforts to accomplish emergent goals, and (3) analyses of children's efforts to appropriate and specialize cognitive forms and functions linked to one practice to accomplish emergent goals in another. Illustrative studies are presented of the model as it is used to understand the development of child candy sellers in Northeastern Brazil.

In this paper, my concern is to sketch a research framework that addresses a problem which cuts across the fields of education, psychology, mathematics, and anthropology: How is it that mathematical understandings that have emerged over the history of a cultural group become the child's own, interwoven with the child's purposive problem solving activities? The problem is addressed in both Piagetian and Vygotskian approaches to cognitive development, but both provide less than adequate frameworks to guide empirical inquiry.

In Piagetian theory, we have a treatment of self-regulative processes that lead the individual to construct novel conceptual understandings; however, the theory offers little insight into the interplay between sociocultural processes and cognitive developmental ones. By reducing cognitive constructions to a small set of universal stages, the framework does not contribute to our understanding of the specificity of cultural forms and the way these forms become interwoven with the cognitive development of the individual.

While Vygotskian constructs of "the zone of proximal development", "scientific/spontaneous concepts", and "semiotic mediation" are inherently more sensitive to socio-cultural dimensions of intellectual development, these constructs are not systematically linked to a larger theory and extended into systematic empirical study.

In this talk, my concern is to sketch an analytic framework that allows us to address questions that reveal the complex interplay between evolving cultural forms and the

constructive activities of the individual in everyday cultural practices.

The Three Component Model

The framework is grounded in a constructivist assumption about cognitive development: as an inherent part of their everyday activities, children construct and address problem solving goals. In their effort to accomplish these goals, children generate new understandings.

Three components are targeted for analysis (SLIDE 1). The first is concerned with the goals that emerge in cultural practices and the way these emergent goals become interwoven with cultural forms like number systems or physical concepts that have emerged over social history. The second is concerned with the interplay between cognitive forms and cognitive functions in cognitive development. Here the concern is how the cultural forms that become interwoven with children's goal directed activities in practices become cognitive ones, serving particular cognitive functions. The third component is concerned with "learning transfer"; here my concern is to understand the way the child may use and adapt cognitive forms generated in one practice to serve new functions in another.

To illustrate these three components, I will draw on my research on the practice of candy selling as conducted by 5- to 15-year-old children in NE Brazil. The candy selling practice is set in the context of a major urban center in Brazil's Northeast in which their is a large informal sector (SLIDE 2). In the informal economy, people sell a wide range of commodities including fruit (SLIDE 3), puffed wheat (SLIDE 4), coffee (SLIDE 5), and, of course, candy (SLIDE 6). Due to a long history of an inflating economy, sellers must deal with very large numerical values in everyday transaction; many of these sellers have little or no schooling.

Component 1: Emergent Goals

To understand the mathematical goals that emerge in the candy selling practice – the first component – requires a four-parameter model contained in Figure 1 (SLIDE 7). The parameters include activity structures, artifacts, social interactions, and the prior understandings children bring to bear on practices. I'd like to consider the way each of these parameters is entailed in an analysis of sellers' practice-linked goals.

Activity structures. Activity structures consist of the general tasks that must be

accomplished in a practice and the general motives for practice-participation. In the case of candy selling, the activity structure is an economic one in which a principal motivation is to generate an income. To accomplish their practice, sellers must purchase their boxes from wholesale stores during a purchase phase (SLIDE 8), price their candy for sale in a prepare-to-sell phase (SLIDE 9), sell their candy in the street in a sell phase (SLIDE 10), and then select new wholesale boxes for purchase in a prepare to purchase phase (SLIDE 11). The cycle then repeats back to the purchase phase.

In selling candy, children generate mathematical goals that are linked to this cyclical structure depicted in Figure 2 (SLIDE 12) to accomplish economic ends. For instance, in the prepare-to-sell phase, one type of mathematical goal that typically emerges is the mark-up from wholesale to retail price. To understand individual's practice linked goals requires an analysis of the problems that must be accomplished in a practice.

Social interactions. The social interactions that emerge in a practice, the second parameter, may simplify some goals and complicate others. Let's take another look at the selling practice as represented in Figure 3 (SLIDE 13). At each phase of the practice, sellers typically interact with other people. For instance, in the purchase phase, sellers purchase their boxes from wholesale store clerks. In these transactions, clerks may offer varied forms of assistance in helping children mark-up their boxes for retail sale. Sometimes this assistance may be merely in the form of telling children how much the box costs if sellers cannot read posted prices. Other times, clerks may tell children what an appropriate mark-up would be to sell on the streets. Regardless, an inherent property of these interactions is that practice-linked problems emerge and are modified in social interactions.

Cultural artifacts. The third parameter consists of the artifacts that are interwoven with the practice. Let's take, as an instance, a pricing convention that has emerged over the history of the practice. In selling their candy, sellers price their candy using a price ratio form. Children might offer their candy to customers for two prices, 3 packages for Cr$500 or 5 for Cr$1000. While this convention may reduce the complexity of arithmetical computations in making change, it may complicate others. For instance, a seller must mark-up a multi-unit wholesale box price in terms of a retail price ratio that reflects the wholesale price plus a profit margin. Thus, the price ratio convention is interwoven with the mathematical goals that emerge as sellers address problems of mark-up.

Prior understandings. The prior understandings that sellers bring to bear on the practice -- the fourth parameter -- are fundamental to the kinds of goals that emerge in a practice. For instance, a seller who does not understand the relation between wholesale price and retail price does not generate mark-up goals -- or at least, if he does, the mathematical goals in his computation will be quite different sort from the seller who does.

•••

In summary, the mathematical goals (SLIDE 14) that emerge in a practice are deeply interwoven with activity structures, social interactions, artifacts, and the understandings children bring to bear on a practice. In the candy selling example (SLIDE 15), we see this quite clearly. The activity structure of candy selling as an economic one in which particular mathematical problems emerge as sellers ply their trade, problems like mark-up computations. The practice-linked social interactions lead to a modulation of sellers goals when sellers make use of assistance from a store clerk or engage in price setting interactions with peers. Conventions and artifacts are interwoven quite centrally to sellers mathematical goals as we have seen in the case of the price ratio retail selling convention. Finally, sellers' prior understandings serve as the basis for their construction of mark-up goals.

In the next two components, the concern is to understand how these emergent goals are interwoven with cognitive developmental processes.

Component 2: A Developmental Analysis of Children's Understandings

In children's efforts to accomplish practice-linked goals, we see an interplay between cognitive form and function in cognitive development. Let's again consider the case of candy sellers; I will draw on an illustration that is prototypical of the interplay between form and function.

Consider the young seller who in transactions with customers makes use of the price ratio -- selling his candy for 3 bars for Cr$1000. The price ratio itself is a cultural form with a distinct social history. For the 6-year-old seller, this cultural form serves the cognitive function of mediating exchanges of candy for currency in seller-customer transactions: The child seller exchanges, for instance, one Cr$1000 bill for 3 candy bars. For the young seller, issues of price mark-up are taken care of by others.

As sellers take on more responsibility, we see a shift in the cognitive functions the price ratio serves. Rather than being solely used in the context of seller-customer transactions, sellers begin to use the price ratio as a means to organize their mark-up computations.

I'd like you to consider how the following 12-year-old unschooled seller arranges his calculation by use of the ratio selling convention -- the convention that sellers initially use to mediate their transactions with customers.

This seller begins his day with a full box of candy bars (SLIDE 16). The box contains 30 units, and he paid Cr$8000 for the box as illustrated in Figure 4. He's selling the bars at 3 for Cr$1000, and he was questioned about how he determined his prices. The child explained that he counted each group of 3 as Cr$1000. Each count of 3 then represented one sale, and he counted two groups of 3 at a time. Thus, he counted the gross price by stating, THESE TWO (2 groups of 3) BRING CR$2000, THESE 2 (2 groups of 3) CR$4000, THESE 2 CR$6000, ... THESE TWO CR$10000. So, the child determined that selling the units for 3 for Cr$1000 on the street would yield a gross of Cr$10000, and subtracting the wholesale price of $8000 from the Cr$10000, determined that he would net Cr$2000. He was questioned about whether this was a good profit and he responded: "IT'S NOT GOING TO BE VERY GOOD. BUT, IF I SELL 2 FOR CR$1000, IT'S GOING TO BE HARD TO SELL, AND IF I SOLD 4 FOR CR$1000, I'D LOSE TOO MUCH."

So, this seller appears to have uprooted the price ratio from its use in the Sell Phase to mediate transactions with customers. Rather than mediating between customer and seller, the ratio is now used referentially -- to refer to such potential customer-seller transactions. It is this symbolic use that allows the seller to recalibrate potential gross prices. Further, this symbolic environment provides the seller with greater opportunities for constructing mathematical relations that emerge in the act of selling. For instance, a seller may discover the relation between increases in ratio size and decreases in the gross value of his box. In the example I just cited, for instance, we see that this seller already understands this mathematical relation -- as the number of units per bill denomination increases, the gross value of the box decreases.

<u>Component 3: The Interplay between Children's Learning across Practices</u>

The third component is concerned with the interplay between form and function across cultural practices. As depicted in Figure 5 (SLIDE 17), in their daily activities

individuals are engaged in multiple practices in which mathematical goals may emerge. So, we can ask a question about "transfer of learning": In what way do cognitive forms elaborated in one practice become appropriated and specialized to accomplish new functions in others? Let's consider a schooled seller's solution to the price mark-up problem and the interplay between math learning at school and in the candy selling practice (SLIDE 18).

This seller (13 years, 5th grade completed) begins his day with a box containing 50 candy bars for which he paid Cr$7000 as depicted in Figure 6. He's selling to customers at 4 bars for Cr$1000. In his mark-up computation, this schooled seller first showed how he determined the wholesale price per unit by multiplying different possible wholesale unit prices times the number of units in the box with a standard school algorithm. First he tried Cr$120 times 50 and then Cr$140 times 50. He then explained that SINCE I SELL 1 CANDY FOR CR$250, I PROFIT CR$110 PER UNIT, subtracting CR$140 from Cr$250 (3d), and that "IF I SELL 4 FOR CR$1000, I PROFIT CR$440 (3e). In order to calculate his net profit for the box, the boy then multiplied Cr$440 (his profit after each time he sold 4 bars) by 12 (the number of potential sales in his 50 bar box using the selling convention 4 bars for Cr$1000) using the standard algorithmic approach and achieved the appropriate product, Cr$5280 (3f).

In the case of this seller, we see clear evidence of transfer. The seller makes use of algorithms learned in school to address problems of the practice. You should also note that the seller's transfer may be most usefully conceived as an extended process of repeated constructions, and not simply as an immediate generalization or alignment of prior knowledge to a new functional context in standard treatments of transfer. In the case of the this seller's solution, the child is using a trial and error multiplication approach to determine an intermediate wholesale unit price value, an approach that may eventually lead to the construction of a more systematic use of a division operation. As in the case of the developmental analysis linked to Component 2, this practice-based model leads to a conceptualization of transfer as an extended process of appropriation and specialization as children repeatedly address problems that emerge again and again in cultural practices.

<u>Concluding Remarks</u>

My aim in this paper has been to sketch a research framework geared towards

understanding the interpenetration of socio-cultural and cognitive developmental processes in children's developing understandings. Through framing questions of development and transfer around the goals that children generate in participation in cultural practices, we come to analyses of cognitive development that are at once richly linked to the constructive activities of the individual and deeply interwoven with socio-cultural life.

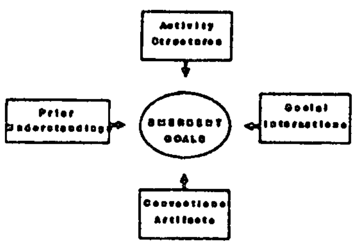

Figure 1. Four parameter model

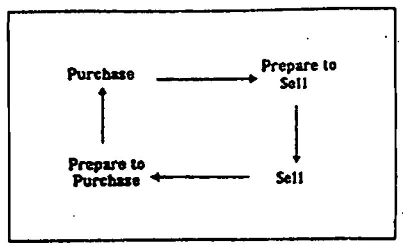

Figure 2. Skeletal structure of the candy selling practice

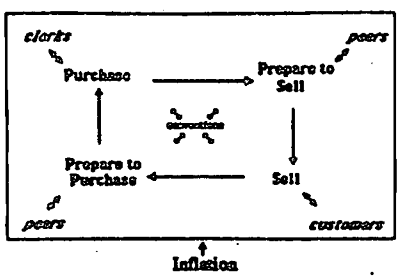

Figure 3. Structure of the candy selling practice

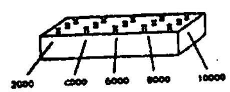

Figure 4. Unschooled seller's mark-up solution

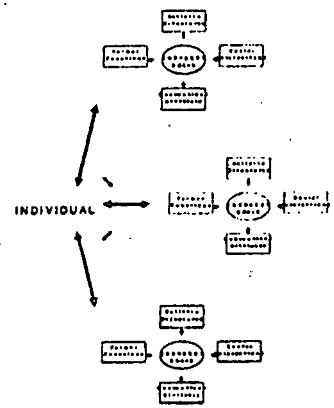

Figure 5. Expansion of four-parameter model

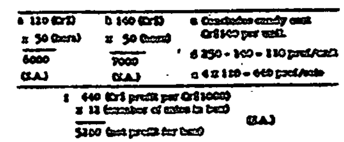

Figure 6. Schooled seller's mark-up solution

TEACHERS' AND STUDENTS' BELIEFS AND OPINIONS ABOUT THE TEACHING AND LEARNING OF MATHEMATICS IN GRADE 4 IN BRITISH COLUMBIA.

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The 1990 British Columbia Mathematics Assessment provided an opportunity to investigate not only students' achievement but also students' and teachers' beliefs and opinions. It included items about mathematics and jobs; about the importance, the difficulty, and the enjoyment of particular mathematics topics; about mathematics classroom practices; and about students' opportunity to learn mathematics. Although teachers indicated they have recently changed their practices, students' responses suggest that use of manipulative materials and use of calculators by students (two practices advocated in the curriculum) are still relatively rare.

In May, 1990 the British Columbia Mathematics Assessment was carried out with the involvement of virtually all students in the province enrolled in Grades 4, 7, and 10 (ages 9-10, 12-13, 15-16 years) and their mathematics teechers. This, the fourth provincial Mathematics Assessment, was similar to three previous ones (Robitaille & Sherrill, 1977; Robitaille, 1981; Robitaille & O'Shea, 1985). It included multiple-choice achievement tests, open-ended tests of problem solving, and questionnaires for both teachers and students. Interest in the results of the 1990 Assessment is increased by the fact that since the 1985 Assessment there has been significant curricular change in mathematics with the introduction of a revised provincial mathematics curriculum (British Columbia Ministry of Education, 1987, 1988). This paper examines results of the Assessment for Grade 4 (Schroeder, in press) focussing on teachers' and students' beliefs and opinions. Its purpose is to give a picture of the teaching and learning of mathematics as seen through the eyes of teachers and students.

Sources of Data. Each Grade 4 mathematics teacher (N = 1980) was given a teacher questionnaire containing a number of general items concerned with the implementation of the revised curriculum and specific questions about mathematics topics, as well as questions about classroom practices used in teaching mathematics and questions about students' opportunity to learn the mathematics necessary to answer particular achievement items correctly.

Nearly every Grade 4 student in the province (N = 39 509) was given an Assessment booklet containing background questions (sex, age, program, and language of instruction); questions about their beliefs regarding mathematics and jobs; questions about the importance, the difficulty, and their enjoyment of topics in school mathematics; and questions about the frequency of various practices in their mathematics classrooms during a typical school week. Each booklet also contained 40 multiple-choice items designed to measure students' achievement in four curriculum strands: Number and Number Operations, Data Analysis, Geometry, and Measurement. The items were selected to reflect intended learning outcomes (ILOs) up to Grade 4 as listed in the revised curriculum guide, and the percentage of items in each strand corresponds closely to the estimated time allotment for that strand. A total of 126 achievement items were distributed across four forms.

Curriculum Revision. The nature and the magnitude of the recent curriculum change in mathematics in British Columbia can be judged by analyzing curriculum documents published by the Ministry of Education (1978, 1987, 1988). The most noticeable differences in the documents are in their size and level of detail. The 1978 guide for Grades 1 to 12 is an 82-page booklet, but the 1987 guide for Grades 1 to 8 is over 400 pages long, and the 1988 guide for Grades 7 to 12 is over 300 pages long. All three guides list specific ILOs, but the most recent guides also include "limiting examples" to clarify the level of development expected, and tables showing estimated time allotments for each strand and grade.

Analysis of the guides also reveals numerous changes in the scope, sequence, and grade placement of topics. In the elementary grades the most substantial changes are in the increased emphasis on decimals earlier in the program, the postponement of operations with common fractions, and the inclusion of a new curriculum strand entitled "Problem Solving Skills." The 1978 guide states that problem solving is an important goal, but in discussing this goal suggests that the problems to be addressed should be routine ones in which situations in the physical world are translated into the language of mathematics, and mathematical solutions are related back to the physical world. By contrast, the 1987 guide specifically includes a range of types of problems (such as application problems, puzzle problems, and open-ended problems), a variety of general problem-solving strategies (such as drawing diagrams, making lists and tables,

looking for patterns, and working backward), and a number of problem-solving processes (such as formulating plans, checking the reasonableness of solutions, explaining solutions, and creating and extending problems).

Another prominent feature of the 1987 elementary curriculum guide not found in earlier guides is a "Statement on Calculators" (p. 7) which indicates that the curriculum was developed "assuming that all students will have access to calculators." It recommends that "at the primary level, the calculator should be used for exploratory activities," and it points out that some ILOs indicate explicitly that students should use a calculator, while other ILOs leave calculator usage to the discretion of the teacher.

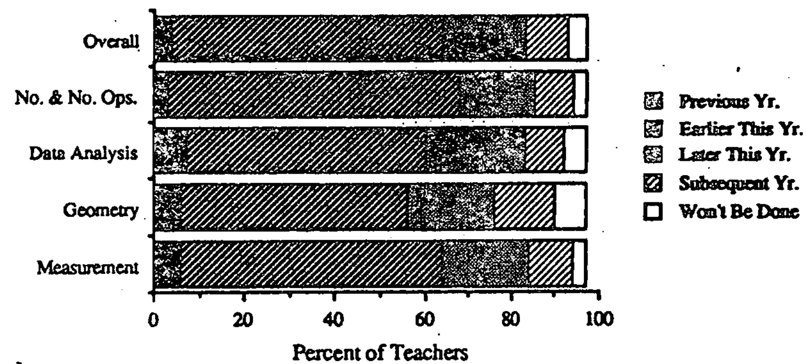
Both the 1978 and the 1987 guides advocate the use of concrete manipulative materials. In the former guide this emphasis is found in an introductory "General Statement" (p. 1) and in an illustrative "Sequence of Activities" (p. 2), but not explicitly in the ILOs. In the revised curriculum this emphasis is found in the Rationale (p. 3) and explicitly in numerous ILOs.

Results for Teachers. To the question, "In the past three years, has your approach to teaching mathematics changed?" 76% of the teachers replied yes and 22% replied no. In a follow-up question, 79% of the teachers indicated that they were now more likely to use cooperative learning groups, 78% were more likely to have students use concrete materials, 78% were more likely to focus on problem solving processes, and 51% were more likely to have students use calculators. To the question, "In the past three years, has your approach to student evaluation in mathematics changed?" 56% replied yes, and 41% replied no. However, 76% of the teachers indicated that they were more likely to evaluate problem solving strategies as well as answers, and 71% of the teachers indicated that they were more likely to assess students through informal observations during class time. While 78% of the teachers agreed or strongly agreed with the statement, "The teaching strategies suggested for the new curriculum (e.g. use of manipulatives, use of calculators) are effective," and 64% of the teachers agreed or strongly agreed with the statement, "The new curriculum helps students learn effective problem-solving strategies," fewer than 5% of the teachers disagreed with either of these statements.

The teacher questionnaires also contained questions about whether students had had the opportunity to learn the mathematics reflected in the achievement items. For each selected item

teachers were asked to chose one of the following statements: "It was done in a previous school year," "It was done during this school year," "It will be done later this year," "It will be done in a subsequent year," or "It will not be done for reasons not listed here." The data collected for 57 student achievement items (out of a total pool of 126 items) are presented in Figure 1.

Figure 1: Opportunity to Learn

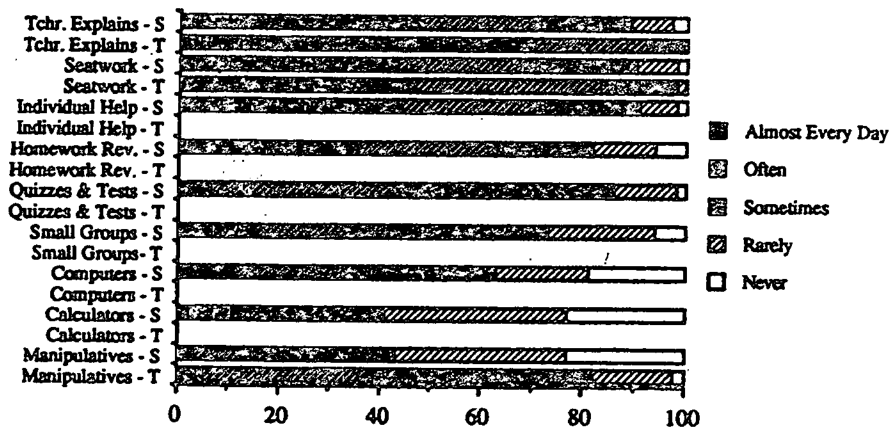


All items were judged by the members of the Assessment team and an item review panel to correspond to ILOs up to Grade 4 listed in the curriculum guide published by the Ministry of Education (1987). Yet averaging over all teachers and over the 57 selected items included in the questionnaires, 15% of the content was rated as "will be done in a subsequent year," or "will not be done." In the Geometry strand, 21% of the content was rated in these categories. It appears that teachers at large have a view of the appropriateness of some curriculum content that differs from that of the curriculum developers and the professionals responsible for the Assessment. That is, a discrepancy exists between the intended curriculum and the implemented curriculum. Furthermore, the Assessment was administered during May at a time when 15% of the school year remained, but 19% of the content was rated as "will be done later this year." This suggests that the teachers have overestimated the extent of the implemented curriculum.

Students responded to nine items dealing with classroom practices; they were instructed to think of their mathematics classes during a typical school week and indicate whether each activity took place "almost every day," "often," "sometimes," "rarely," or "never." The teacher questionnaires contained three classroom practices items very similar to three of the student items, but the remaining six student items were not presented to the teachers as a result of an

error in printing. The items and the teachers' and students' responses are shown in Figure 2. Comparison of teachers' responses with students' responses on the three common items suggests that teachers generally rated all of the classroom practices as occurring more frequently than students rated them. Furthermore, teachers almost never said that they "never" use the practices mentioned, although some students did select "never."

Figure 2: Frequency of Classroom Processes



The most striking discrepancy between teachers' and students' reports of classroom practices is seen with respect to the use of manipulative materials. Students responded to the statement "We use objects like blocks, counters, and geoboards," while teachers responded to "Students use objects like blocks and counters." The proportion of teachers who said they use manipulatives "almost every day" or "often" corresponds roughly to the proportion of students who responded in these categories or "sometimes," while the segment of teachers who use manipulatives "sometimes" corresponds roughly to the students who use them "rarely," and the group of teachers who use manipulatives "rarely" or "never" corresponds to the proportion of students who "never" use them.

Results for Students. Students' responses to the items presented in Figure 2 suggest that most Grade 4 mathematics classes are traditional in nature involving review of homework and teacher lecture followed by individual seatwork with help from the teacher as needed.

However, variations on this pattern are also seen. About one-third of the students indicated that they work in small groups "almost every day" or "often," about the same number indicated that they do so "sometimes," and about as many indicated that they "rarely" or "never" do so. Surprisingly, a similar distribution of responses was found with respect to computer usage. This is in marked contrast to the results for Grade 7 students, only 5% of whom reported using computers frequently in their mathematics classes and 85% of whom "rarely" or "never" used computers. Despite strong curricular support for calculators, nearly three-fifths of the students indicated that they "rarely" or "never" used them in their mathematics classes. Although both the 1978 and the 1987 curriculum guides emphasize the importance of experience manipulating concrete materials, student reports of classroom practices suggest that concrete materials are seldom used; 55% of the students responded that they "rarely" or "never" use objects like blocks, counters, and geoboards; only 12% responded that they do so frequently.

Students also responded to 12 multi-part items regarding topics in school mathematics and how important they felt each topic was, how easy they found each topic, and how much they liked each topic. Five-point scales ranging from "not at all important" to "very important," from "very difficult" to "very easy," and from "dislike a lot" to "like a lot" were used. Results from these items are presented in the table below.

Topic	Important/ Not Important	Easy/ Difficult	Like/ Dislike
Adding, subtracting, multiplying, and dividing whole numbers.	89/2	64/16	74/11
Learning about decimals.	81/4	63/15	67/12
Learning about fractions.	80/4	66/13	68/11
Learning how to estimate.	77/7	70/13	68/13
Learning things about geometry like shapes, flips, turns, and slides.	75/8	65/15	71/13
Checking answers.	86/4	77/9	66/15
Using graphs.	71/6	59/17	62/15
Learning about place value.	78/4	56/18	58/16
Learning about measuring weight, height, length, and width.	82/6	63/17	68/14
Learning how to use calculators.	75/12	85/5	81/6
Using objects such as blocks, counters, and geoboards.	53/18	70/9	56/20
Learning strategies for problem solving, such as looking for patterns and making models.	76/7	49/26	64/15

Students' responses to this group of items indicate that most students feel that all the listed topics in school mathematics are quite important and that the traditional topics are of the

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greatest importance. Operations with whole numbers is given the highest importance rating; the lowest importance rating is given to using manipulative materials. The majority of students find these mathematics topics easy; learning to use calculators was rated easiest. Learning problem solving strategies was rated as the most difficult. The majority of students expressed liking for the topics listed. In general, students tended to like the topics that they find easy more than those that they find difficult, but the relation between difficulty and liking is not an especially strong one. One topic that is an exception to this generalization is using manipulatives, which was rated relatively easy but was the least liked. This topic was also given the lowest importance rating.

Students' beliefs about and attitudes toward mathematics and jobs were explored in three items which asked whether they agreed or disagreed with the following statements: "You have to able to do mathematics to get a good job when you grow up;" 85% agreed (or strongly agreed), 5% disagreed (or strongly disagreed); and "Most people use mathematics in their jobs;" 82% agreed, 6% disagreed; and "When I leave school, I would like a job where I have to use mathematics;" 54% agreed, 18% disagreed.

Detailed analyses of correlations between students' achievement and their attitudes, beliefs, and opinions have shown that students with positive attitudes (e.g. those responding that mathematics is important, or easy, or enjoyable) generally scored 5% to 10% higher than students with negative attitudes. For example, students who rated learning about decimals important outscored those who rated decimals unimportant by 43% to 32% on items involving decimals. Correlations between achievement and classroom practices revealed that where a clear majority of the students reported that the practice occurred either frequently or seldom, the achievement of the majority was higher than the achievement of the minority. For example, the 55% of the students who indicated that they seldom "use objects like blocks, counters and geoboards," scored an average of 54% correct, while the 12% who indicated that they use such materials frequently scored 48% overall. Similarly, the 56% of the students who responded that they seldom use calculators scored 54% correct overall, while the 11% of the students who indicated that they use calculators frequently scored 48% overall. Where there was no consensus regarding the frequency of a practice, differences in achievement were small.

Discussion and Conclusions. Teachers indicated that they have made substantial changes in their mathematics teaching. More than 75% said they have changed their approach and are now more likely to teach and evaluate problem solving processes, to use concrete materials, and to have students work in cooperative learning groups; about half are more likely to have students use calculators. Correlation analyses suggest that teachers may be more likely to have low-achieving students use manipulatives and calculators than high-achievers.

Students rated learning problem solving strategies as moderately important, but most difficult, and relatively little liked. About a third of the students said they work in small groups frequently, and about the same number do so sometimes; small group work may be the context in which problem solving is often done. Some interesting discrepancies exist between teachers' and students' responses. For example, although 36% of the teachers said they use manipulative materials frequently and 20% said they rarely or never do so, only 12% of the students said they use manipulatives frequently while 55% said they rarely or never do so. Students gave their lowest importance rating to manipulatives, but rated them relatively easy. Surprisingly, they also gave manipulatives their lowest enjoyment rating, perhaps indicating that they perceive their use not to be an integral part of mathematics and hence a waste of time. Although half the teachers reported that student use of calculators is increasing, nearly 60% of the students said they rarely or never use calculators in their mathematics classrooms.

References.

British Columbia Ministry of Education Curriculum Development Branch. (1978). *Mathematics Curriculum Guide: Years One to Twelve.* Victoria, BC: Author.

British Columbia Ministry of Education Curriculum Development Branch. (1987). *Mathematics Curriculum Guide 1-8.* Victoria, BC: Author.

British Columbia Ministry of Education Program Development Branch. (1988). *Mathematics Curriculum Guide 7-12.* Victoria, BC: Author.

Robitaille, D. F. (Ed.). (1981). *The 1981 British Columbia Mathematics Assessment: General Report.* Victoria, BC: Ministry of Education.

Robitaille, D. F. & O'Shea, T. J. (Eds.) (1985). *The 1985 British Columbia Mathematics Assessment: General Report.* Victoria, BC: Ministry of Education.

Robitaille, D. F. & Sherrill, J. M. (1977). *The British Columbia Mathematics Assessment: Test Results.* Victoria, BC: Ministry of Education.

Schroeder, T. L. (in press). Grade 4 results. In D. F. Robitaille (Ed.), *The 1990 British Columbia Mathematics Assessment: General Report.* Victoria, BC: Ministry of Education.

PROBLEM SOLVING AND THINKING: CONSTRUCTIVIST RESEARCH

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This paper describes the evolution of the research design of the Problem Solving and Thinking Project. An earlier description of our research as evolutionary was presented in Budapest (Schultz, 1988) at the International Congress on Mathematics Education. In that paper we discussed the element of recursion in our research as it facilitated the design of an optimal environment for data collection and relevance to the classroom teacher. The present paper presents our thinking three years later of the emergent research design in the Problem Solving and Thinking Project.

The Problem Solving and Thinking Project, 1986-1990, sponsored by the National Science Foundation, generated knowledge which contributed toward a newly-funded Teacher Enhancement grant called the Atlanta Math Project, 1990-1994. This subsequent project, a four-year implementation of the National Council of Teachers of Mathematics Standards (1989), like the former, operates with a constructivist view of learning, teaching, and research. It is already producing compelling evidence that changing beliefs and behaviors in the mathematics classroom is occurring.

The final products of the Problem Solving and Thinking Project are being reported three fold. First, a paper is to be published on the teacher education model (Schultz & Hart, 1990) that emerged from the project. Second, a summary of data analysis results (Schultz, 1991) will be presented at PME-NA this fall at Blacksburg, VA, USA. Finally, the present paper is offered as an attempt to describe the constructivist nature of the design of the study. Initial thinking of our research

as constructivist was presented in Budapest at ICME (Schultz, 1988). In that paper we discussed the element of recursion in our research in our attempt to design an optimal environment for data collection and to make the research relevant to the classroom teacher. The present paper is an account of the influencing factors in the development of our current thinking on this emergent or constructive nature of the research design in the Problem Solving and Thinking Project. We believe this sharing is useful to the international mathematics education research community since the Problem Solving and Thinking Project is in the mainstream of research in the United States.

Background

The Problem Solving and Thinking Project investigated the relationship between middle school inservice teachers' metacognitive activity and knowledge and their problem-solving ability. The improvement of teachers' problem-solving abilities through the Project focused on a metacognitive and constructive process. We assumed a relationship between metacognitive activity and mathematical problem-solving performance; specifically, that the monitoring and regulation of one's knowledge, beliefs, and strategies impact problem solving.

At the writing of our original proposal, however, we were influenced by our reviewers and critics' conventional research leanings, which meant that we needed to describe what we wanted to do more in hypothetico-deductive terms than in constructivist terms. Therefore, the statement of the problem was given in complete and clear language with a predetermined

plan of our investigation. We articulated how we would obtain our data and then analyze it. "Subjects," as we called them, would be as representative and random as feasible given that teachers had to make application for "selection" into our Institute, the forum for the study. We argued for videotaped interviews, teaching, and problem-solving protocols along with more conventional instrumentation as problem sort tasks and problem-solving tests. We described how we would triangulate quantitative with qualitative data.

Though our proposal writing and our beliefs were to some extent discrepant, having obtained the funding, we assumed a trusting confidence existed in us as researchers. And so, step by step we made our critical decisions as the project unfolded believing that good research is honest research.

The Focus of the Project

At the time of our original proposal, we shared with PME-NA in Columbus, Ohio, USA (Hart & Schultz, 1985) our interest in knowing if an individual's metacognitive activity can be increased through instruction which focuses on metacognitive experience and knowledge. And, we asked if problem-solving success can be improved through increased metacognitive experience and knowledge. After some experience with the data, we told PME at Montreal, Quebec, Canada (Hart, 1987) that we realized we did not know whether the right questions were being asked. A new question emerged to define the territory of the inquiry: What categories of phenomena associated with individual problem-solving performance and construction of

mathematical knowledge can we identify and how do those domains interact with metacognition? This appeared to be the legitimate and broader-based focus of our investigation. At the same PME meeting, we explained (Schultz, 1987) that the simultaneous influencing of multiple factors made it impossible to sort a single causality such as metacognition. Therefore, we were prepared to study it in relation to other factors. The problem became more fluid, the questions more tentative, and our reactions more comfortable. The upshot of this is that determining the research question <u>a priori</u> presupposes a significant assumption: that the problem situation is understood so well that precision in determining research questions is assured.

Instrumentation

For 10 weeks in 1987, 15 teachers from elementary, middle, and secondary schools in four school systems in the metropolitan Atlanta, Georgia area enrolled in the Institute on Problem Solving and Thinking, a five-hour graduate course through Georgia State University. These teachers taught at least one mathematics course at the middle school level. The Institute centered around model/experience/reflect and facilitate/experience/reflect recursive sequences of activities by the teacher educators (Schultz & Hart), the teachers, and their students. Videotaping was liberally used to record and reflect on the problem-solving protocols of each group.

The research approach was qualitative, grounded in naturalistic inquiry (Lincoln & Guba, 1985), and reliant on the technique of episodic parsing of protocols (Hart & Schultz,

1985; Schoenfeld (1983). Critical decision-making junctures occurred weekly after each class while we (Hart & Schultz) engaged in reflective debriefing sessions--much like we had our teachers do after a problem-solving period or after each class. It was during these discourses that the detailing of our research design emerged. (Just as in the natural classroom setting, where assignments are collected and graded and new plans for the next class are made.) We reviewed (analyzed) videotape data, teacher reflection logs, test results, and field notes. We then reconsidered our goals and objectives for the Institute and designed the next classes' events. Occasionally, we shared our interpretations of data with the teachers as a class or as individuals to negotiate language used to describe our observations. We functioned within the interpretive frameworks of teacher and teacher educator, also referred to as partners in research. What followed in the Institute classes--our forum for teaching and learning as well as data collection and analysis--was mutually valued and meaningful to teachers and teacher educators. We became partners in teaching, learning, and researching.

In the end, we had data from the teachers in the form of: (a) pre and post videotaped problem-solving protocols of individuals and small groups, (b) pre and post videotaped problem-solving lessons taught by the teachers, (c) videotaped segments of teachers' respective problem-solving, (d) written reflection logs, (e) pre and post problem-solving tests, and (f) problem-solving sort tasks. Data were collected from the teacher educators in the form of : (a) videotaped individual

problem solving, (b) videotaped paired problem solving, (c) refection logs, and (d) videotaped model teaching. From the students of the teachers, data were collected in the form of (a) pre and post problem-solving tests and (b) pre and post problem-solving sort tasks.

Analysis

It can be seen from the foregoing that constructions and reconstructions of our research focus, techniques, and analysis were neither linear nor did they produce readily quantifiable information. Instead, these events occurred contingently, contiguously, at times virtually simultaneously, and most often qualitatively.

At the conclusion of the Institute and for some months thereafter, there was much left to do with the substantial amounts of data obtained. Each teacher agreed in writing to be available for the next year or two to respond to our requests to review further analysis and continuing interpretations of their data. Collaborative university graduate internships, educational specialist scholarly papers (Ropp, Villani, and Wingfield), doctoral dissertations (Najee-ullah and Lee), and independent and co-analysis efforts by the researchers themselves were conducted within a constructivist framework for negotiating research design with research goals.

Results

First of all, an outcome of the study was a model for mathematics teacher education. We look forward to sharing this through publication. This model is already successfully in

place in the earlier mentioned Atlanta Math Project. Secondly, the researchers found the emerging research questions and design (the subject of this paper) to make possible the results of the data analysis below.

Two kinds of opposite directional beliefs emerged: productive and nonproductive. For example, "It's okay if doing math takes time" is an expressed productive belief; whereas, "I'm good in math if I can do the problems fast" is an expressed nonproductive belief. In general, it was initially found that middle school mathematics teachers exhibited more nonproductive beliefs than productive beliefs during their own problem-solving protocols. Moreover, it was found that in the brief 10-week course beliefs could be reconstructed through the influence of the inservice teacher education we offered. All of the commonly-held beliefs were nonproductive before participating in the teacher training with a positive correlation between nonproductive beliefs and problem-solving ability. However, the number of subjects who expressed productive beliefs increased after the course with a higher correlation with successful problem-solving ability.

In closing, we called the Problem Solving and Thinking Project "constructivist research," where new knowledge was constructed through the interpretive frameworks of the Problem Solving and Thinking Project participants and teacher-researchers. Ironically, we attribute the success of the project's emergent knowledge to the tenacity with which we held onto our framework of constructivist teaching, learning, and research.

References

Hart, L. (1987) Interactive domains, metacognition and problem solving. In J. Bergeron, N. Herscovics, & C. Kieran (Eds.), Proceedings: PME, Vol. II (pp. 236-242). Montreal, Quebec, Canada: University of Montreal.

Hart, L., & Schultz, K. (1985). Metacognition research: Techniques of protocol analysis. In S. K. Damarin & M. Shelton (Eds.), Proceedings: PME-NA, (pp. 106-111). Columbus, OH.

Lincoln, Y. S., & Guba, E. G. (1985). Naturalistic inquiry. Beverly Hills: Sage.

Schoenfeld, A. (1983. Episodes and executive decisions in mathematical problem solving. In R. Lesh & M. Landau (Eds.). Acquisition of mathematical concepts and processes (pp. 345-395). New York: Academic Press.

Schultz, K. (1988, July-August). Recursion in mathematics teacher education. Paper presented at the Sixth International Congress on Mathematics Education, Budapest, Hungary.

Schultz, K. (1991). Problem solving and thinking: Report of findings. Paper to be given at the meeting of PME-NA, VA.

Schultz, K. (1987). Theoretical perspectives on inquiry into problem solving, knowledge acquisition, metacognition, and teaching. In J. Bergeron, N. Herscovics, & C. Kieran (Eds.), Proceedings: PME, Vol. II (pp. 250-256). Montreal, Quebec, Canada: University of Montreal.

Schultz, K., & Hart, L. (1990) An experiential teacher education model for reflective mathematics teaching. (Unpublished manuscript, Georgia State University, Atlanta.)

Assessment of thought processes with mathematical software

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The assessment of skills has often been based on multiple choice tests, enabling to evaluate general levels of ability only. Microworlds are seen as tools which can enhance particular skills, and promote problem solving strategies. However, the assessment of these educational goals is generally neglected. Our goal in this paper is to show through an example, that the microworld itself can provide tools for the assessment of thought processes occurring during problem solving sessions.

1 Qualitative assessment of aptitudes versus ongoing tracking of actions: the role of open ended software

Traditionally educational testing has been based on quantitative assessments of individuals' general levels of ability and achievement relative to others within a group. The tools have often been multiple choice paper-and-pencil tests assessing the performance of a particular skill, often without addressing the questions of previous knowledge, strategies in problem solving and mental models or misconceptions. Frederiksen and White (1990) stressed the role that open ended software and intelligent tutoring systems can take in an ongoing assessment of acquisition of knowledge, in particular during problem solving sessions. However, generally mathematical open environments are not used to evaluate acquisition of skills. Rather, they are presented as a series of tools which permit hopefully to attain pedagogical goals. The evaluation is often global or is restrained to a simple description of activities done with the system. Moreover, in this evaluation the role of the environment

is either absent (when the test is done by paper and pencil), or confined to be a technical tool only. Our aim is to present a case study of the evaluation of processes of thought with a mathematical environment, the Triple Representation Model, and to show how different kinds of tests can complete each other in their analysis.

2 Description of TRM

The Triple Representation Model (TRM) is a microworld about functions. It is described and accompanied by numerous examples of activities in Schwarz, Dreyfus and Bruckheimer (1990). We will present here TRM from the more general perspective of its tools and of its educational goals.

The tools are typical to mathematical open ended software:

- **Multiple representations:** Three representations of functions (algebraic, graphical and tabular) are available. Within each of them, simple actions are automatically carried out (reading a graph, computing images, managing data.....)
- **Control level:** The user can consult or transport the data gathered in a particular representation from one representation to another.
- **Manipulation of high level abstractions:** In each of the representations, the functions are manipulable objects: they can be graphically magnified, shrunk or stretched; algebraic manipulations are transformations of the type $x \dashrightarrow f(x+h)$ or $x \dashrightarrow |f(x)|$.
- **Programmation:** the user can study an algebraic condition within a range of numbers. TRM displays for which numbers this condition holds. Typical conditions are $f(x) < a$, $f(x) = g(x)$, $f(x) > f(x+e)$ or more generally $f(x) < g(f(x))$. This is called the Search operation.

The educational goals are:

- **Link between representations:** information collected within a representation is meaningful in others.

- **Conceptual thinking:** the general goal was formulated as richness of the concept of function; more specifically, we expressed this goal as: a) relying on linear approximation to estimate values of functions between two points; b) knowing that an infinity of functions can pass through two given points; c) linking ordered series of values of preimages to ordered series of images by a functional relation. These three characteristics are qualitative and express a concern for functional thinking in mathematics (b and c) as well as in experimental sciences (a and c).
- **New strategies:** Use of several representations to solve problems; manipulation of graphs ("zoom in", shrink....) in order to discover properties; more generally use of the open-ended character of TRM to guess, test, check and modify hypotheses in problem solving sessions.
- **New mathematical resources:** study of the properties of a wide range of families of functions (greatest integer, rational functions, square roots, absolute value and composite of such functions).

We will concentrate here on the role of TRM in the evaluation of the nature of links between representations and of aspects of functional reasoning.

3 Description of the assessment tools

Around TRM was created a set of activities which covered an introductory approach to functions including basic concepts like domain, range, (pre)-images, properties like maximum, domain of increase, basic skills like reading constructing and interpreting graphs and tables, modeling algebraic rules, and study of a various range of functions (linear, square-root, polynomials, greatest integer....). The study of all these subjects was not systematic; most of the activities were open-ended; work of pairs of students on the computer changed with teacher-led class discussions.

We used TRM with 3 classes of 9 graders during 12 weeks. During this process some subjects were given a TRM task (***CINDAT***) whose goal was

to measure some aspects of functional reasoning. This task is described in Schwarz & Bruckheimer (1988). Succinctly, the child is given a rectangle and a hidden function for which (s)he can compute automatically as many images as (s)he wishes. The task consists of deciding whether the function passes through the given rectangle or not. This is always a non-trivial decision given that the (hidden) functions were of the following types (see Figure 1):

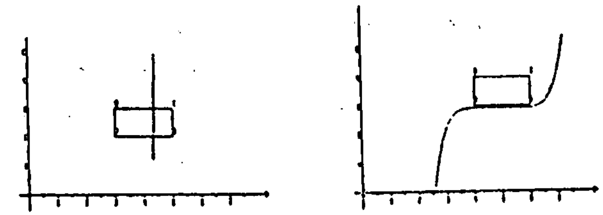


Figure 1: (Hidden) functions for the CINDAT task

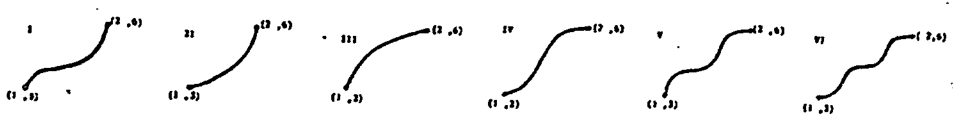
Such a task provided interesting behaviors ranging from a decision that the two functions do not pass through the rectangle because all the trials "fall outside", to subtle considerations linked to linear interpolations.

At the end of the 12 weeks, the three classes were given a ***Questionnaire*** whose goal was to assess certain aspects of link between representations (by comparing the compatibility of information given in two different representations), aspects of functional thinking (series of (pre)images), and a strategy that was new for the students (manipulation of graphs to discover properties of a function). The questionnaire was a multiple choice test. However, students were asked to justify their answers. One of the questions (which will be analyzed below) was:

A function f satisfies:

f(1)=3	*f(1.1)=3.05*	*f(1.2)=3.1*	*f(1.3)=3.15*	*f(1.4)=3.25*	*f(1.5)=3.60*
f(1.6)=4.20	*f(1.7)=5.60*	*f(1.8)=5.90*	*f(1.9)=5.97*	*f(2)=6*	

Which of the following graphs is the "best fit" for f?



Finally, the subjects were interviewed while solving an ***OPEN BOX problem*** with TRM. The precise formulation of the task is given in Schwarz and Dreyfus (1989). It is a maximum problem which demands an extensive use of the Search operation and encourages the use of multiple representations. Our goal was to investigate the links students make between representations and between a series of preimages and images (functional reasoning) during the problem solving process.

The Questionnaire, the OPEN BOX task and CINDAT represent extremely different methodologies for the evaluation of similar goals. We will see that the OPEN BOX task (done within TRM) provides the richer results.

4 Assessment

We will concentrate here on only two aspects of the assessment: the link between representations and one aspect of functional thinking (link between series of preimages and images).

Students' justifications to the question from the Questionnaire presented above are of four different categories:

- Correct verbal justification of the kind: "First the function increases moderately, then climbs quickly, and then it slows down again", without graph or computation (49%).
- Construction of a graph; correct comparison with the six graphs (12%).
- Computation of rates of change; correct comparison with the graphs (7%).
- Correct answer without justification or wrong answer (32%).

There are two alternatives for the interpretation of the first category: a)

students "see" graphs through numerical data; b) students compare graphs and numerical data. The first eventuality demands skills which go far beyond a simple transfer of information. However, the second possibility is also demanding: the solver has to read and interpret a series of data given numerically and to compare it to an interpretation of parts of the graphs (the comparison can also be done from the graph to the algebraic data). The conclusion that at least half of the TRM students could successfully transfer information from one representation to another one was corroborated by other questions asked in the Questionnaire. However, it was based on an indirect analysis of the answers.

The OPEN BOX task provided the possibility of an inspection of the frequency, the amount, the nature and the quality of transfer of information. Typically students worked for about 30 minutes on this task, and performed about 20 microworld operations. Our aim was to observe the construction of links between different functional representations during this problem solving process. For this purpose, a complex assessment methodology was needed: Each student's sequence of operations was recorded and interpreted according to the following scheme:

Two indices were defined, the transfer index T and the quality index Q. The transfer index shows how many times a student switched from one representation to another one during the session, and at how many of these transitions (s)he had made full use of all the information obtained previously. Sample indices are listed in Table 1.

Student	T	Q
SS	(2, 2+, 0-)	0.8
DC	(4, 4+, 0-)	0.6
MY	(3, 2+, 1-)	0.25
NV	(4, 2+, 2-)	0.05

Table 1: Four student's indices

For example, MY's transfer index T=(3, 2+, 1-) shows that her solution

process had four stages, say first a graphical one, then a numerical one, then another graphical one, and finally an algebraic one; thus there were 3 transitions; at two of these (2+), all the information gained in previous stages was taken into account, at one of the transitions (1-) some information was lost or forgotten.

The quality index shows at which average rate a student progressed toward the solution.. For each operation carried out by the student (whether or not the representation was changed), an after/before ratio R of variation of distances from the solution was formed. A value of R close to 1 expresses optimal progress, a ratio of 0 expresses no progress at all. The quality index is the average of all these values over all the student's operations. MY's $Q=0.25$ expresses an overall mediocre rate of progress. In fact the sequence of MY's actions provides a series of quality indices. For a precise definition of T and Q , see Schwarz (1989).

Good transfer indices show successful transfer of information between representations. But only in combination with a high quality index could we be certain that such successful transfer of information was meaningful for the student. We thus imposed the following strict criterion: transfer index of the type (n, n+, 0-) and high quality index ($Q>0.6$). Out of the 42 students in the sample, 23 satisfied this strict criterion among them SS and DC. It was interpreted that for these students the representations complete each other, reifying some properties of functions. Nine other students, among them MY, made more successful than failed transfers and had a mediocre quality index, $0.2<Q<0.4$; they were interpreted as seeing properties of functions as attached to representations, being unable to grasp these properties as objects on their own. Seven students, among them NV, with either low quality index or bad transfer index were analyzed as not interpreting well properties of representations of functions.

The OPEN BOX task provided also a very interesting fact concerning functional thinking: many students used intensively and successfully the Search operation without explicitly using any graphical referent. They could interpret, during the problem solving session, a series of preimages and images and find the location of the maximum by successive converging searches. This conclusion was in accordance with the behaviors with CINDAT: such students tried to compute images on the basis of linear interpolations and took into consideration the set of all their attempts to draw a conclusion. They could "see" through their numerical data a whole picture of the function under consideration and interpret it in a graphical way although no graph was presented.

Problem solving sessions with a TRM provided conclusions about processes of thought involved with the link between representations and the relation between a series of images and preimages. Such conclusions could not be drawn from paper-and-pencil questions (even if explanations were asked). Therefore the role of educational software is double: a) to promote of educational goals; b) to enable the educationist to assess these goals.

Bibliography

Frederiksen, J. & White, B. (1990). Intelligent Tutors as Intelligent Testers. In N. Frederiksen, R. Glaser, A. Lesgold & M. Shafto (Eds), *Diagnostic Monitoring of Skill and Knowledge Acquisition* (pp. 1-25) Hillsdale, NJ: Laurence Erlbaum Associates.

Schwarz, B. (1989). *The Use of a Microworld to Improve Ninth Graders Concept Image of a Function: the Triple Representation Model Curriculum*. Doctoral dissertation, Weizmann Institute, Israel.

Schwarz, B., & Bruckheimer, M. (1988). Representations of functions and analogies. In *Proc. 12th PME Conf.*, pp. 552-559, Veszprem, Hungary.

Schwarz, B., & Dreyfus, T. (1989). Transfer between function representations: a computational model. In *Proc. 13th PME Conf.*, Vol. 3 pp. 269-276, Paris.

Schwarz, B., Dreyfus, T., & Bruckheimer, M. (1990). A model of the function concept in a three-fold representation. *Computers and Education* 14, No. 3, pp. 249-262.

SPONTANEOUS STRATEGIES FOR VISUALLY PRESENTED LINEAR PROGRAMMING PROBLEMS

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Integer linear programming problems were presented, in a visual format, to tenth graders who had not yet learned any solution methods for such problems. Seven groups of solution strategies, which the students developed spontaneously, were identified and classified as visual, algebraic or mixed. Students' choice of strategy was found to depend on the problem type: The more directly they were asked for a verbal description of a strategy, the more likely they were to choose an algebraic one; in applied problems, all students used visual strategies.

Introduction

Many mathematical problems can be solved either visually or algebraically. For example, a linear programming (LP) problem can be solved using visual imagery of the target function as part of a hyperplane over the admissible domain; it can also be solved algebraically, e.g. by the simplex method (Danzig, 1960), in which matrix manipulation is needed.

According to Presmeg (1986), "A visual method of solution is one which involves visual imagery, with or without a diagram, as an essential part of the method of solution, even if reasoning or algebraic methods are also employed" (p. 298). We will use this as definition of a visual method for solving a problem, and define an algebraic method accordingly as one that involves algebraic calculation as an essential part. Consequently, mixed methods, in which visual imagery and algebraic calculation are both used in an essential way, may also exist.

We will use the terms *solution method* and *strategy* interchangeably. A strategy is a general scheme for solving a set of problems. Following Presmeg, strategies may be classified as visual, algebraic, or mixed. It was our aim to investigate visual, algebraic, and mixed strategies, which were spontaneously generated by students for solving visually presented problems. Therefore, we needed a set of problems which (i) were of a type unknown to the students, (ii) could easily be presented in visual format, and (iii) allowed for a variety of visual and algebraic solution strategies. Two-dimensional LP problems satisfy those demands. A two-dimensional LP problem requires maximizing the value of a linear target function

with two independent variables, under constraints consisting of a set of linear inequalities. An example is presented in Figure 1.

> A candy factory produces two kinds of candy. To produce 3kg of lemon candy, 1kg lemon and 2kg sugar are used. To produce 3kg of bitter-lemon candy, 2kg lemon and 1kg sugar are used. The factory has 150kg sugar and 100kg lemon. The profits are \$2.00 for 1kg of lemon candy and \$2.50 for 1kg of bitter-lemon candy. What is the most profitable production?

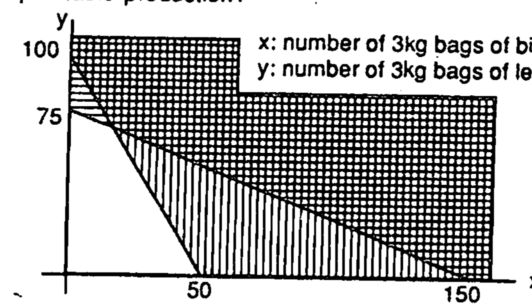

Figure 1: A two-dimensional LP problem.

Only integer two-dimensional LP problems will be considered in this paper; these are problems in which the two variables can take on integer values only and the variation is therefore restricted to a grid. Because of the relative simplicity of such problems, it was easier for students to develop solution strategies.

The learning environment

Our aim was to identify strategies that were spontaneously generated by students with little prior instruction and who had not yet learned how to solve LP problems. For this purpose, students were to be put into a problem solving situation, which was novel for them, and were required to devise ways to deal with the problems presented. Therefore, we needed an environment conducive for developing solution strategies. Moreover, this environment needed to allow for easy transition between visual and algebraic representations, so that the generated strategies could be visual or algebraic. For these reasons, a computerized environment was specifically designed and developed.

This computerized environment consists of six games for two players. All the games have a similar structure: A two-dimensional integer LP problem is presented verbally and graphically. The target function is not known to the players. The game board is the visual presentation of the problem (Figure 2a). The two players start

from the same grid point on the board (the point (3,3) in Figure 2). Each can move, in turn, up, down, left or right to a neighboring grid point, if this neighboring point satisfies the constraints of the LP problem. After each move, the value of the target function at the chosen point appears. The winner is the one who first reaches the optimal point (Figure 2b).

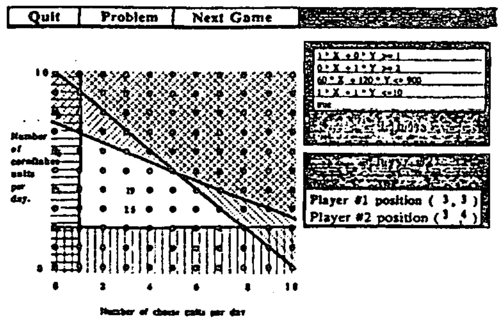

Figure 2a: The computerized environment after the first move.

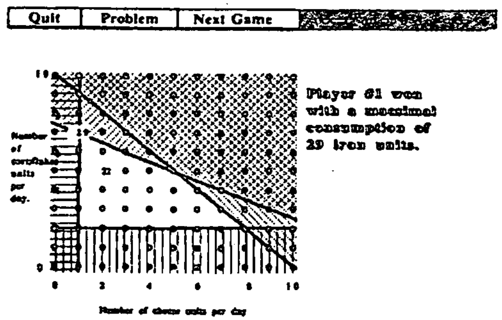

Figure 2b: The computerized environment at the end of the game.

Method

The research was carried out with 49 average ability tenth-graders who had not previously been taught linear programming. For the purpose of identifying their spontaneously generated strategies, we followed the procedure developed by Case (1980), which has three phases: Observation of subjects solving a task spontaneously, formulation of hypothesized strategies, and observation of subjects solving tasks modified in such a manner as to discriminate between the different hypothesized strategies.

The first phase was carried out in a pilot study. Students were briefly introduced to LP; they were shown what types of problems are amenable to LP, but care was taken not to mention any solution methods. Students were then presented with LP problems in the computerized learning environment; with few exceptions, they worked in pairs. Their solution paths were observed in detail and supplemented by their answers to a questionnaire. In the second phase, strategies were hypothesized on the basis of the data collected in the first phase. An open paper-and-pencil questionnaire was developed for the third phase; questions were written in light of the hypothesized strategies found in the pilot study. Following Case, the questions were written in such a manner that different hypothesized strategies would yield different solution paths.

Both verbal and applied questions were included. A verbal question is one which calls for the description of a strategy, such as: "What is the best strategy for winning the game?". An applied question is one where the subject is asked to solve a simple problem (and thus apply a strategy). Questions intermediate between verbal and applied ones were also included; for example, a grid point in the interior of the admissible domain was given and subjects were asked to explain why at this point the target function could not obtain its maximal value.

The following generalizations of integer two-dimensional LP problems were also included as applied questions: problems where a minimum rather than a maximum value of the target function was required, continuous problems, problems with non-linear constraints, and three-dimensional problems.

The questions were not only aimed at gathering data on the development of spontaneous strategies, but also at finding which strategies are most frequently used, and which strategies are preferred in which situations.

Strategies

The strategies we identified were classified as being visual (V), algebraic (A) or mixed (M).

Visual strategies for solving LP problems are strategies, which use a visual representation of the constraints and the target function as an essential part of the solution process. Following Noelting (1980), the visual strategies, that we found, were further classified according to the degree of interaction between the

dimensions of the problem. Noelting described three degrees of interaction in the solution of two dimensional problems: In the first degree, one dimension of the problem is considered exclusively, as if the second one did not exist at all. In the second degree, both dimensions enter the subject's consciousness, but one of them is chosen at the beginning and all further action concentrates on that one dimension; which dimension is chosen depends on the situation. In the third degree, the two dimensions are integrated. In LP problems, the dimensions are the variables of the target function: horizontal and vertical moves in the computerized game.

The following visual strategies were found, in correspondence with Noelting's three degrees:

(V1) Strategies which identify as optimal an extreme point in vertical or horizontal direction, whereby the choice between vertical and horizontal direction is always the same. For example, one subject gave the following strategy description: "We should try to get to the highest point".

(V2) Strategies which identify as optimal an extreme point in vertical or horizontal direction, whereby the choice between vertical and horizontal direction is based on a comparison of the corresponding changes in the target function value. Example: "I'll move one step up, and one step to the right, and observe in which direction the number grows most, if at all, and I'll move in that direction".

(V3) Strategies which identify as optimal an extreme point in a direction which is not necessarily vertical or horizontal: For each V3 strategy, one can design a problem such that the solution provided by the strategy is not an extreme vertical or horizontal point. V3 strategies were further classified into strategies choosing the direction of the gradient of the target function (V3b) and strategies choosing an arbitrary direction in the quadrant in which the value of the target function grows (V3a).

Strategies were classified as algebraic, if they were based on a formula for the target function. Such a formula was not provided by the computerized environment; but it could be constructed from the changes in value of the target function after horizontal and vertical moves on the game board. Two algebraic strategies were found:

(A1) Constructing a formula for the target function and comparing its values at all points satisfying the constraints, i.e. the interior and the boundary points of the admissible domain. Example: "I will check for each point in the limits what will be the value of the formula, and then I'll see what is the largest value".

(A2) Constructing a formula for the target function and comparing its values at the vertices of the admissible domain.

Mixed strategies consist in conjecturing some points as possibly optimal on the basis of visual considerations, and deciding which one of them is the optimal point by algebraic computation. Two mixed strategies were found:

(M1) Different visual strategies may lead to different optimal points. The choice between these is done by comparing the corresponding values of the target function.

(M2) This strategy starts from a point on the boundary of the admissible domain, suspected by a visual strategy as being optimal, and continues by moving along the boundary, comparing the value of the target function in neighboring points until it stops growing.

All visual strategies identified in this research are incomplete: They may give correct answers to some problems, but will fail on others. In Figure 3, an example is given in which each of the above visual strategies identifies a different point, none of which is the optimal one.

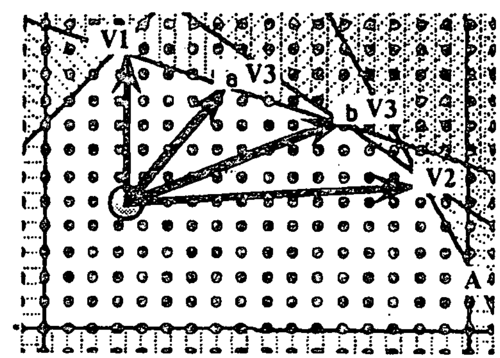

Figure 3: Wrong answers from identified visual strategies; Target function is f(x,y)=4x+y; optimal point is A.

Both algebraic strategies will yield the correct answer, if the admissible domain is a (bounded) polygon; (A1) proceeds by brute force; (A2) is an algorithm often taught in high school; it is based on a theorem stating that the optimal point is a vertex. (M2) is a simple version of the simplex algorithm. It will yield the correct answer, if the admissible domain is a convex polygon.

Results and Conclusions

Visual, algebraic and mixed strategies were identified in this study; while visual strategies were most frequent, in some questions more than one third of the subjects chose algebraic strategies. It can thus be concluded that for visually presented problems, both visual and algebraic strategies are likely to be developed.

Visual strategies were further classified according to the degree of interaction between the dimensions. Purely one-dimensional strategies (of type V1) were most frequent. Analoguous findings, for pre-school age children, were reported by Wilkening & Lange (1990). We conclude that visual strategies, spontanously generated by highschool students for solving a new type of problem, are not more sophisticated than those generated by younger children.

Among the visual strategies, the higher ones were used less often than the lower ones: (V1) was used more frequently than (V2); few subjects used (V3). For mixed strategies, the situation was reversed: Only 23% of the subjects used mixed strategies at all, but of these, 70% used (M2); all of these started with (V3) to reach a point on the boundary; thus, although (V3) was rarely used by visual solvers, it was relatively frequent among subjects using mixed strategies. Subjects who used mixed strategies of type (M1) started from points suspected as optimal on the basis of visual strategies (V1) and (V2). Thus low visual strategies were used together with low mixed strategies, high visual strategies together with high mixed strategies, although no a priori relationship between these strategies existed.

As should be expected, most subjects were not consistently using one same strategy. But there were patterns: Verbal problems tended to sollicit algebraic strategies; visual strategies were favored in applied problems. In order to simplify the discussion, we will from now on exclude subjects who have used mixed strategies; such strategies were anyway used predominantly for intermediate problems. In the most verbal question, 60% of the subjects described algebraic strategies, while in some applied questions only visual strategies were used. Subjects consistently behaved according to the principle: The more verbal the question, the more algebraic the strategy: For instance, all subjects who used a algebraic strategies in intermediate questions described algebraic strategies in verbal questions. Thus subjects could be classified into three types: Visual solvers, visual-algebraic solvers and algebraic solvers: Subjects who described a visual strategy in a verbal problem were classified as visual solvers; subjects who chose

algebraic strategies for intermediate and verbal questions were classified as algebraic solvers. No subject consistently chose algebraic strategies in applied problems. The behavior of the three solver types is summarized in Table 1.

Question / Solver type	Applied	Intermediate	Verbal
Visual	V	V	V
Visual - Algebraic	V	V	A
Algebraic	V	A	A

Table 1: Strategies used by different types of solvers.

In conclusion, we note that no signs of avoidance of visual strategies were observed in this study, in which problems were presented visually and no strategies (neither visual nor other ones) were previously known to the subjects. Quite the contrary: Visual strategies were developed more frequently than either algebraic or mixed ones. This leads us to believe that avoidance of visualization may be a learned effect! It should, however, also be noted that correct visual strategies for linear programming problems are not evident, even in the two-dimensional case; and therefore all of our subjects' spontaneously developed visual strategies were incomplete: Visual does not always mean easy!

References

Case R. (1980). Intellectual development and Instruction: A Neo-Piagetian View. In Lawson A. (Ed.), *The Psychology of Teaching for Thinking and Creativity.* (p. 59-97).

Noelting G. (1980). The Development of Proportional Reasoning and Ratio Concept. Part I - Differentiation of Stages. *Educational Studies in Mathematics, 11*(2), 217-255.

Presmeg N. C. (1986). Visualization and Mathematical Giftedness. *Educational Studies in Mathematics, 17*(3), 297-311.

Wilkening F. & Lange K. (1990). When is Children's Perception Holistic? Goals and Styles in Processing Multidimentional Stimuli. In Globerson T. & Zelniker T. (Eds.). *Cognitive Style and Cognitive Development.* Norwood, N. J.: Ablex.

INITIAL DEVELOPMENT OF PROSPECTIVE ELEMENTARY TEACHERS' CONCEPTIONS OF MATHEMATICS PEDAGOGY

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Prospective elementary teachers views' of mathematics pedagogy were examined to characterize initial development of their ideas. Analysis of their written responses to a hypothetical pedagogical task resulted in a set of hypotheses about their ideas. They believe that students need to be actively involved although they do not have a well developed model of student learning. They are believe that understanding is important and supported by visualizations. They have difficulty decentering from their own thinking to focus on students' thinking.

In recent years, views of mathematics educators have been converging to provide a vision for mathematics instruction (NCTM, 1989 & 1991; NRC, 1989). According to this vision, mathematics learning is a process of active individual construction by the learner and social construction by the classroom mathematics community (Cobb et al., 1991) which involves formal and informal modes of thinking and solution and concrete and abstract levels of representation (Kieren, 1988). Mathematics is a problem solving activity, a language for describing aspects of the world, and a subject of wonder and interest to be studied for its own sake. Mathematics is to be created and understood by learners not just memorized. It is essential that learners develop their ability to communicate mathematical ideas. The teacher's role is one of setting contexts for encounters with problematic mathematical situations and of facilitating small and large group work and communication.

This vision of mathematics instruction is a significant departure from the views that seem to underlie current elementary school mathematics instruction (Cobb, Yackel, Wood, & McNeal, in press). It is the responsibility of teacher education programs to begin the preparation of elementary school teachers to participate in this new vision. As a result, we must call into question previously held ideas about the preparation of mathematics teachers.

Currently mathematics teacher education is limited by an underdeveloped knowledge base with respect to teacher development. How might we create appropriate models for research on mathematics teacher development? One answer might be to adopt by analogy methods that have been successfully used to study students' development of mathematical concepts. The goals of mathematics education and of mathematics teacher preparation are parallel in several ways. Both aim to develop conceptual understanding, not just procedural facility; to encourage autonomous problem solving; and to promote communication of powerful ideas. What can research in mathematics teacher education learn from research in mathematics education?

Mathematics education researchers have examined closely the conceptual understandings of learners at different stages of development with respect to the concepts of interest (e.g. multiplicative structures, concept of limit). They have attempted to characterize the knowledge and beliefs of students, to note patterns in the sequence of development, and to uncover early concepts that appear resistant to change. In this study, we have attempted to begin to apply such methodology with respect to teacher development.

The goal of this study was to begin to describe the conceptions of prospective elementary teachers (PETs) early in their preparation to teach. What ideas are readily developed or changed and which are not developed or are resistant to development in initial stages of learning to teach. The traditional views of prospective teachers as they begin teacher education programs have been documented[1] (Ball, 1989). The vision of the possible elementary teacher has been described (NCTM, 1991). However, little literature exists on the development of teachers between these two extremes. (It should be noted that this is not a study on the effectiveness of a particular intervention.)

Subjects

The study focused on the thinking of PETs in their final year of study at our university. These PETs were enrolled in a mathematics education course on the teaching and learning of mathematics in the elementary school. The course was designed to challenge their limited views of mathematics and traditional views of mathematics learning and teaching and to assist them in constructing views more consistent with the views described in the first section of this paper.

Students participated in this course four hours per week for ten weeks. The first five weeks of the course involved the PETs in mathematics lessons which promoted their construction of mathematical ideas and in analyzing student thinking using video-taped interviews of elementary school children. The second five weeks were devoted to developing questioning skills and to designing instruction. A ten week course is a minimal intervention. It allowed us to focus on the PETs' initial development.

Methodology

Methodology for assessing teachers conceptions of mathematics learning is still in its primitive stages. The pedagogical task used in this study represents the researcher's conviction that inferring beliefs about learning from teachers' engagement in pedagogical tasks is more reliable than asking teachers what they believe about mathematics learning. In the latter case there is a strong risk that the teachers will demonstrate language which they acquired in their courses rather than the schema which will likely direct their pedagogical decisions.

The following pedagogical task was given to the PETs to complete in writing at the end of the ten-week course:

> Imagine that you are teaching multiplication of fractions. One of your students raises his hand and says, "I am very confused. This doesn't make sense. The answers I am getting are smaller than one of the numbers I started with! What am I doing wrong?" He shows you the following example that he has done.
>
> $$2\frac{1}{4} \times \frac{2}{3} = 1\frac{1}{2}$$
>
> What would you as a teacher do? Identify what you would say/do in response to this question. Include all teaching behaviors that would be stimulated by

this interaction.[2]

The task is a scenario that the PETs might face in the classroom. It was chosen because it involves conceptual knowledge, provides some information about the learner, but requires the PET to define the pedagogical problem.

Forty PET responses were randomly selected from two of the sections of this six-section course. Using a phenomenographic approach (Marton, 1988), data was identified from which PETs' conceptions of mathematics pedagogy could be inferred. These data were sorted into categories which were modified several times to better represent the data. From these categories, two sets of hypotheses were produced, one which characterized aspects of PET development beyond a traditional view of mathematics pedagogy and one which characterized difficulties in their ability to generate instruction consistent with current visions. Although there are clearly differences between subjects' conceptions, these hypotheses represent an attempt to characterize aspects that are common to a majority of those involved in the study.

Results and Discussion

Because of space limitations, one representative example[3] of PET work is presented and commented on, followed by a summary of the hypotheses made in this study. (Additional samples of PET work will be included in the oral presentation.)

A part of Sara's work on the task follows.

> I would present the student with the equations such as one times one or one times three. He would then demonstrate understanding by manipulating the blocks into equal sets. Therefore, I would say that he has conceptual understanding of multiplication. I would then ask him to derive a generalization about one times another number. His generalization should be that one times another number equals itself.
>
> Next I would have the student work with the rods and cubes to create 2 1/4. The manipulative would provide him with a concrete visualization. Then I would ask him what would 2 1/4 x 1 equal. Drawing on his previous knowledge he should say that the answer is 2 1/4. He would then be given instructions to draw two equal rods.... He would then be asked to leave one as a whole and separate the second into thirds. He would then shade in the one whole rod and shade in 2/3 of the rod divided into thirds. I would then ask him to think about which rod was greater. I would then probe him to find out what he could tell me about 2/3 compared to 1. He should answer by saying 2/3 is less than 1. This would illustrate his understanding that 2/3 is less than 1.
>
> Next I would refer back to the student's generalization that one times another number will equal the same number. Based on this generalization I would ask the student to hypothesize what would happen if he multiplied a number less than one such as 2/3 times one. He would then state a hypothesis such as the answer would be 2/3. We would then test his hypothesis to see if it were true. First he would draw a rectangle. Then I would ask him to show me how he would divide the rectangle into thirds. He would then go through the problem again using the semi-concrete example to show that the solution of the problem would be 2/3. This activity would allow him to understand that 2/3 is less than 1. In this way he will be able to move on to the original problem and more fully understand that when 2/3 is multiplied times 2 1/4 a smaller answer is given.
>
> He and I would then go back to the original problem. He would demonstrate his understanding of 1/4 by manipulating two rods and a cube. I would then explain that since the problem asks to multiply 2 1/4, 2 1/4 is considered a whole set not 2 wholes and 1/4. I would then ask him, if 2 1/4 is a whole and it is to be multiplied by 2/3 how many equal parts must 2 1/4 be divided into.... The student would then divide the whole into parts. This semi-concrete diagram allows the student to apply his understanding of the concepts presented thus far.
>
> After the student draws the diagram, I would have him multiply 2 1/4 x 2/3. He may shade in the diagram to check or prove his answer. In this way, he will visually recognize that 2/3 of 2 1/4 equals 1 1/2. Therefore, he will have reinforced the concept presented previously by applying it to the original problem.

Sara's solution to the pedagogical problem demonstrated an approach which differed from the traditional "teacher as teller"[4] in a number of ways. She revealed a commitment to aiding the student to understand. She made extensive use of concrete and diagramatic representations and posed questions to the student much more than she told him information. Her strategy seemed to be based on an awareness that failure to understand a problem may be a result of insufficient knowledge at a more basic level and that mathematical understanding must be built on prior knowledge.

One notices early in this sample that Sara seems to have in her mind a sequence to the thoughts through which the student will proceed (beginning from 1x1 and 1x3). Rather than presenting a problematic situation for which the student would have to marshall his resources and reorganize his knowledge, she leads him through a sequence of steps. In essence, the student (as in a lecture) must follow her thought sequence as contrasted with a teacher who attempts to follow the student's thoughts and to stimulate a growth in his ideas. Notice that almost all of her questions require only a very short response. We could characterize her model of teaching as "if I can get him to produce a particular sequence of answers, this will lead to understanding of the final step." A mechanism that involves student construction of ideas in response to problematic situations seems to not be central to her model.

We would argue in this problem that the student is unable to visualize $2\frac{1}{4} \times \frac{2}{3}$. Therefore, his development of a visualization for the multiplication of fractions is likely to be a key step in his sense-making for this expression. In Sara's scenario, however, the development of this visualization is not emphasized; rather the conclusions drawn from the representation are emphasized.

In the final steps of her scenario, we can also observe that the student's original question, why is the product greater than one of the factors, is not directly addressed. There seems to be an assumption that if the student sees the computation represented visually that a general understanding of the problem will result.

All of the discussion of Sara's solution applies to the solutions of many of the PETs. While solutions to the pedagogical problem posed varied among individual PETs, hypotheses emerged which characterized a large percentage. Their solutions can be characterized as follows.

1. They interpreted the student's correct answer and subsequent question as evidence of algorithmic competence but without understanding.

2. They showed their commitment to fostering understanding.

3. They recognized that other students probably could benefit from instruction on this matter.

4. They believed that understanding is the result of student's activity and conversation, not just listening and watching.

5. They considered the use of manipulatives and other means of visualization to be helpful in the development of understanding.

6. They believed that understandings must be built on prior knowledge.

Three factors appeared to limit their solutions to this pedagogical problem.

1. an inability to focus on the mathematical knowledge that would resolve the student's confusion,

2. an inability to decenter from their own thought processes in order to focus on the thinking of the student, and

3. an inability to see the development of understanding as a process of individual construction initiated by a problematic solution.

Each of these is considered in a bit more depth below. These factors are not disjoint sets; there is undoubtedly considerable overlap and interaction between them.

1. Inability to focus on the mathematical knowledge that would resolve the student's confusion. Many of the PETs responded to the student's question in one or more of the following ways:

a. proving the correctness of the answer using manipulatives or diagrams (seen in Sara's approach)

b. having the students compute many such products so they could see that the product was consistently less than the larger factor, and/or

c. developing a pattern of products in which one of the factors is held constant and the other is decreased showing that the product also decreases.

These strategies support the appropriateness of the answer computed by the student, but fail to explain why the product should be smaller than one of the factors.

My conjecture with respect to these teaching strategies is that the PETs involved are in the early stages of developing what it means to offer a conceptual (mathematical) explanation. Most of their prior experiences focused on procedural explanations. When a conceptual question is posed, they often do not know what would constitute an answer to the question. (Our research currently in progress[3] seems to provide some evidence to support this conjecture.)

2. Inability to decenter from their own thought processes in order to focus on the thinking of the student. This difficulty was seen in two ways:

a. Lessons began at levels that were extremely basic without any evidence that work at such a level was needed (Sara began at 1x1).

b. Lessons followed the teacher's thought processes not the students with key ideas imposed on the student (Sara's lack of attention to the development of a visualization for multiplication of fractions).

3. Inability to see the development of understanding as a process of individual construction initiated by a problematic solution. Three types of evidence supported this hypothesis:

a. PET's told or showed students key pieces of the understandings.

b. Representations were introduced without sensitivity to how students develop representation (same example as for 2b).

c. Students were often asked to respond to a series of leading questions as opposed to a series of problematic situations (Discussed for Sara's solution above).

Conclusions

The ideas of the PETs in this study are believed to differ from a narrow transmission of procedures view of mathematics pedagogy because of the mathematical experiences in which they participated during the course. They were not taught to teach in a particular way, but rather were encouraged to make sense of the mathematics learning experiences that they had in the course. Their responses to the pedagogical task give some indications of what initial development might look like. Our results suggest that these PETs have a sense that the students need to be actively involved although they still do not have a well developed model of student learning. They believe that understanding is important and strongly supported by visualizations. Visualization was probably a particularly salient and positive aspect of their learning experiences in the course. Finally, we see as we have seen with inservice teachers (Simon and Schifter, in press), that teaching strategies (e.g. questioning, use of manipulatives) are more easily learned than are new models of students' mathematical learning.

In interpreting these results, the question must be asked, "To what extent is the development described by this study unique to the particular course in which the students participated?" This can only be answered definitively by studies in other similar contexts (i.e. courses with similar goals). It seems reasonable to expect that, given mathematical experiences consistent with current reform efforts (MAA, 1990; Alibert, 1988), PETs will more readily develop some ideas about mathematics learning than others. If this is the case, we might see somewhat consistent examples of initial learning across different but similar educational experiences. Future studies could also be enhanced by observing teachers in actual pedagogical situations which would provide data beyond what is available from hypothetical pedagogical tasks. Also, multiple tasks or "events" (Lesh, 1990) need to eventually be created to get the fullest picture of PET's knowledge development.

Finally, an additional limitation of such work is poorly defined conceptual fields in teacher education. While mathematics educators have struggled to develop domain definitions (Romberg, Lamon & Zarinnia, 1988; Vergnaud, 1983), the key concepts in mathematics teacher education are relatively uncharted.

Footnotes

1. Our research in progress (Construction of Elementary Mathematics (CEM) Project begun in August, 1990 under a grant from the National Science Foundation) has revealed that PETs entering our program hold an almost monolithic view of teaching mathematics as explaining to the students how to do the computation.
2. This task was adapted from a task used by the National Center for Research in Teacher Education at Michigan State University.
3. Sara's work was chosen because it illustrated most of the hypotheses that were made from the data as a whole.
4. CEM observations of PETs in a teacher education mathematics course suggest that this is a key issue.

Bibliography

Alibert, D. (1988). Codidactic system in the course of mathematics: How to introduce it? Proceedings of the twelfth international conference of Psychology of Mathematics Education, Hungary.

Ball, D. (1989). Breaking with experience in learning to teach mathematics: The role of a preservice methods course. Paper presented at the annual meeting of the American Educational Research Association.

Cobb, P., Wood, T., & Yackel, E., Nicholls, J., Wheatley, G., Trigatti, B., & Marcella, P. (1991). Assessment of a problem-centered second grade mathematics project. Journal for Research in Mathematics Education, v.22, pp 3-29.

Cobb, P., Yackel, E., Wood, T., & McNeal, B. (in press). Classroom mathematics traditions: An interactional analysis> In C. Maher & R. Davis (Eds.), Relating schools to the reality of mathematical learning. Englewood Cliffs, NJ: Prentice Hall.

Kieren, T. (1988).Personal knowledge of rational numbers: Its intuitive and formal development. In M. Behr & J. Hiebert (Eds.) Number concepts and operations in the middle grades. Reston, VA: National Council Teachers of Mathematics.

Lesh, R. (1990). Personal communication.

Mathematical Association of America (MAA, 1990). A call for change: Recommendations for the mathematical preparation of teachers of mathematics (Draft).

Marton, F. (1988). Phenomenography: A research approach to investigating different understanding of reality. In R. Sherman & R. Webb (Eds.), Qualitative research in education: Focus and methods. London: The Falmer Press, 141-161.

National Council of Teachers of Mathematics (NCTM, 1989). Curriculum and evaluation standards for school mathematics. Reston, VA: National Council of Teachers of

Mathematics.

National Council of Teachers of Mathematics (NCTM, 1991). Professional standards for teaching mathematics. Reston, VA: National Council of Teachers of Mathematics.

National Research Council (NRC, 1989). Everybody counts: A report to the nation on the future of mathematics education. Washington, DC: National Academy Press.

Romberg, T., Lamon, S., & Zarinnia, E. (1988). The essential features of the mathematical domain of ratio and proportion. Madison, WI: National center for Research in Mathematical Sciences Education.

Vergnaud, G. (1983). Multiplicative structures. In R. Lesh & M. Landau (Eds.), Acquisition of mathematics concepts and processes. London: Academic Press.

THE EFFECTS ON STUDENTS' PROBLEM SOLVING BEHAVIOUR OF LONG-TERM TEACHING THROUGH A PROBLEM SOLVING APPROACH.

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Abstract

Students from two Year 9 classes at an Australian high school were interviewed as they worked on various mathematical problem solving questions. One class had for three years been taught by a teacher very committed to teaching through a problem solving approach and to demonstrating the everyday usefulness of mathematical ideas. The other class had received traditional instruction, with only occasional problem solving tasks given mainly for amusement or as a "fill-in." Three principal differences in their approaches to the questions were observed. Students from the class which emphasised problem solving worked more deliberately and kept helpful written records of their work. They were noticeably less prone to close quickly on an answer by combining numbers in the question in a superficial way. Instead they were more likely to use a guess and check strategy effectively.

Introduction

Although there is a substantial body of research into teaching mathematical problem solving, almost all of the research evaluates the success of programs designed outside the school (sometimes by a teacher-researcher) and therefore taught to children as something special, not as a routine part of the curriculum (see, for example, Charles and Lester, 1984; Groves and Stacey, 1988; Isaacs, 1987; Kantowski, 1977; Stacey, 1989). Until recently, there was little alternative to this: instruction-related research into mathematical problem solving had to begin in curriculum development by researchers, because the teaching of problem solving was not widespread in schools. However, recent changes in curriculum in Victoria (Australia) have encouraged many teachers to develop their own "problem solving approach" to teaching or special programs in their classrooms. This paper examines the effects of one such teacher-devised mathematics program oriented to improving students' problem solving performance by comparing the problem solving behaviour of those students with that of similar students who had had "traditional instruction".

There are important reasons for looking at problem solving programs which have been constructed and delivered by ordinary teachers, which will, after all, be the programs which almost all children receive. These programs can

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persists with the context, gradually drawing the mathematics out from it and introducing the abstract, general language along with the context-specific words. For example, we observed a lesson where he introduced some ideas about linear equations to his Year 9 class by asking the class to determine a taxi's flag-fall and the charge per kilometre given the fares for various journeys. He examined students' naive approaches sympathetically, eventually adding the graphical and algebraic approach. The class then went on to practise plotting graphs, reading intercepts and interpolating and extrapolating values. For most of the lesson, he used the context-specific terms, rather than abstract terms such y-intercept or gradient. Frank is keen to develop in students a questioning attitude and a resourcefulness so that *"if they can't solve it one way, they should be able to solve it another way"*.

Although Frank adopts the "problem solving approach" described above to most of his teaching, he also gave 9P work on "unfamiliar" problems (examples are given in Figure 2) which could be solved with strategies such as drawing a diagram and guess and check. Frank's students were generally positive about this particular component of their work. When we observed one of these lessons, Frank handed out a worksheet of problems similar in length and difficulty to those shown in Figure 2 and suggested that students to work on it in pairs. He helped students as they worked and presented solutions on the board, with interaction from the class. No alternative solutions were discussed and the problems were not extended in any way. "Guess and check" and "make a table" were the only strategies mentioned in this lesson, although he does also stress "drawing a diagram" and "looking for patterns". Frank's program does not contain features broadly associated with "looking back" or increasing meta-cognitive awareness that are widely recommended in the pedagogical literature (e.g. Lester, 1989).

Selection of Tasks and Interview Methods

In order to observe differences in approach to problem solving, we interviewed and audio-taped a stratified sample of 9 of the 24 students from each of 9P and 9N on two occasions, firstly as they worked on the Plumber's Fees (see Figure 1) and then on a selection of 9 questions provided by Frank as being typical of the tasks he used in his problem solving component (see Figure 2). Some prompting was given when students were not making any progress. To encourage students to think aloud, two or three students, roughly matched in ability, were interviewed together: in some groups, students interacted strongly, others not. Frank's problem solving questions were definitely familiar to all his students (and to some students from 9N) but there was no reason to believe that

be expected to differ in a number of ways from the researcher-designed programs which have previously been studied. The goals of a teacher may be different to those of a researcher who is steeped in theories of problem solving. Because of their intimate involvement in the dynamics of the classroom, teachers will also choose goals which they see as achievable in that setting and teaching methods which will enable them to maintain a working atmosphere. Teaching approaches that spring from a teacher's own commitment are the ones that will be carried through consistently over a long time and therefore have a better chance of making a tangible difference.

Selection of Subjects

Stratified samples of students from two classes of Year 9 students at a suburban high school were interviewed. The two classes, known as 9P and 9N, were chosen because they were apparently similar mixed-ability classes from the same cohort of the same school yet with quite different instructional histories. For their three years at the school, the classes had been kept intact as mathematics groups. Class 9N had been taught for three years by teachers who use a traditional approach working from the textbook or worksheets which explain how each type of question is done. Although the class has occasionally been given "problem solving" tasks to do, there has been no regular part of the curriculum devoted to it. One girl , interviewed as she worked on a variant of a missionaries and cannibals crossing a river problem, commented that she had done something like it before "as a sort of joke".

The other class, denoted here as 9P, had been taught for the same three years, by one teacher, here called Frank. During his three years at the school, Frank had become acknowledged as the initiator of problem solving activity in the school and 9P was the class where he trialled his ideas. Frank is particularly concerned with improving the problem solving skills of his students in the widest possible sense and to make them able to deal with everyday applications of mathematics. This, for Frank, is the most important aspect of his teaching. He dates the strong emphasis on problem solving in his teaching from a staffroom conversation, when 9P were his year 7 class.

> *"I was sitting in the staffroom one day when an experienced maths teacher asked me how the classes were going. I explained that I wasn't enjoying it, that the algebra was a real struggle and that the kids were not getting anything out if it. "Well, don't do it", she said. "Teach something useful". That's when the penny dropped. I'm supposed to be some sort of professional. I should be able to make those decisions. Not that I'm not going to do algebra, but I try and direct the course to real life applications - what the kids really need".*

From that time on, Frank began to introduce special problem solving activities in each of his classes and also began to spend more time on the everyday applications of the mathematical content he was teaching. Frank likes to introduce each new topic with a practical example, which he lets students explore in their own ways. He

one class was better prepared for the Plumber's Fees than the other as both had worked on linear equations recently. The question had been selected independently and before Frank's lesson on taxi fares was observed.

Figure 1: The Plumber's Fees.

A plumber's fee is made up of a fixed charge for making a visit as well as a charge depending on how long the job takes. For a job taking 15 minutes, her total charge is $28. For a job taking 40 minutes, her total charge is $58. And for a job lasting one hour, her total charge is $82.
(i) What would be her total charge for a job lasting 70 minutes?
(ii) What is the fixed fee charged by the plumber?
(iii) What is her charge for each minute of work?

Figure 2: Examples of Frank's questions with typical solutions from 9P

1. Which number between 1 and 150 when multiplied by itself produces the closest number to 300?
2. At 6.30am the first two people arrived at the Melbourne Cricket Ground to buy tickets for the grand final. Every 25 minutes after that, 3 more than the number of people already present arrived to get in line. How many people were in the line at 9.00am?

Q1 25×25 = 625 nah!
= 15×15
= 225 maybe
17×17
= 289
18×18
= 324 nah!

Q2
6·30 → 9·00
= 2·30
2hrs30 / 25
150/25
= 6

0 2
1 5
2 8
3 11
4 14
5 17
6 20

20 people at 9·00

Results

As would be expected from their greater familiarity with his questions, Frank's students obtained more correct answers to his problems than did students from 9N. The interviewers' impression is also that the 9P students needed less prompting to solve the Plumber's Fees question. However, the principal purpose of the study was to observe differences in behaviour and the three features described below stood out.

(a) Methods of Recording.

There was an obvious difference in the way that students in 9P and 9N approached the problem solving tasks. Whereas all but one or two of the 9P students worked fairly methodically, students from 9N rushed and guessed, frequently punching the buttons of their calculators furiously. Despite the fact that ample scrap paper and graph paper was supplied , only one student from class 9N wrote anything at all (except for the answers) during the fifty minutes they worked on Frank's questions. The lack of an organised written record was a clear cause of lack of success for 9N on several occasions. In contrast, students in class 9P kept a much more helpful written record at both problem solving sessions, (although their records were neither neat nor sequential) and moreover, they used their recording as a guide to help them decide what to do next.

(b) Grabbing at surface relationships

As explained above, the students of 9N attacked problems in a rush. This behaviour seems to be a symptom of their relentlessly trying ways of manipulating the numbers in the question to come up with an appropriate answer. This tendency to grab at surface relationships rather than to explore the problem was the major mathematical difference between class 9P and class 9N. With the Plumber's Fees problem, all four 9N groups arrived at a (wrong) answer within a few minutes. In three of these groups, students quickly agreed that the answer was $114, which is obtained by noticing that 70 can be made up as 40 + 2 x 15 and so deducing that the price for 70 minutes is made up from the price for 40 minutes plus twice the price for 15 minutes. This method seeks the wanted quantity from a simple combination of the data in given in the question. Only one of the 9N students saw his first answer of $114 as a hypothesis, rather than a definite answer. He went on to test the consistency of his hypothesis with the data provided. For all the others, this early closure on a wrong solution clearly blocked their subsequent thinking about the question, even though they were told that the initial answer was wrong.

The solutions from students in 9P began quite differently. No student suggested $114. Two students were able to solve the problem quite directly: one with a graph and another tabulated differences and found a complicated but correct number pattern. The others had a period of exploring the problem (either guessing a fixed charge, a charge per minute or calculating ratios of prices to times) but they did not close quickly on an answer. Five students used a "guess and check" strategy and some features of this behaviour will be taken up later.

A similar tendency to grab at surface relationships was in evidence in the solutions by 9N to Frank's questions. For example, the wording of Question 2 is particularly hard to interpret. Most of the students in 9P (and two of the three 9N students who were finally prompted to make a table) were unable to co-ordinate all aspects of the question and consequently obtained (wrong) answers of 20 or 77. A typical solution from 9P is shown in Figure 2. Students from the below average group of 9N exhibited quite different "grabbing relationships" behaviour. Their first, almost immediate, solution was 125 people - *"every 25 mins after 6.30, 5 (3 more than 2) people arrive so 25 mins times 5 people is 125 people"*. One of the three students disagreed with this and set about finding the number of 25 minute intervals in the two and a half hour period from 6.30 to 9.00. This was done by calculating 2.30/25 (= 0.092), 2.30/0.25 , 150/0.25 and 150/ 0.992 (presumably an error for 0.092). The group then multiplied 9.2 by 5 which gave 46, their agreed answer to the question.

(c) Use of "guess and check"

Along with the use of tables and charts, "guess and check" is the principal strategy emphasised in Frank's approach to problem solving. The wording of Frank's question 1 (see Figure 2) for example, perhaps triggers a "guess and check" approach and reflects Frank's emphasis. All but one of the 15 students who tackled this at interview, solved it by "guess and check", choosing whole numbers and multiplying them by themselves on the calculator until the answer was nearly 300. Apart from the inefficiency caused by absence of written records of the trials by 9N, there was no difference observed in the effectiveness with which the two classes of students were able to reach an integral solution.

At the simple level required by Question 1, guess and check is an intuitive strategy available to all, but the Plumber's Fees brought out substantial differences between the groups. Table 3 gives the numbers of students from each class who attempted The Plumber's Fees and the three of Frank's nine questions which were amenable to a guess and check strategy. Also given in Table 3 are the numbers who used a guess and check strategy either largely successfully or unsuccessfully.

The lack of spontaneous choice of guess and check by 9N is probably explained by their urge to close quickly on an answer. Since superficially manipulating numbers is "successful" (in that answers were produced), they did not spontaneously seek any other methods and the interviewers' preferred prompts were graphical and algebraic

methods. Frank's students (9P) instead began by exploring the question (by calculating differences, ratios or guessing) and in this way, five of them were lead into a guess and check solution. The four successful students used less than three guesses each to find the fixed charge or rate. The two pairs of 9N students who unsuccessfully used guess and check after prompting, failed. Onepair did not appreciate the logical structure of a guess and check solution, the other lacked an adequate written record. In order to find the fixed charge by guess and check, students have to realise that the resulting charge per minute will be constant at the three data points. Without this guiding understanding of structure, the method makes no sense. In contrast, the two instances of failure of guess and check by students of 9P were more technical in nature. One student was unable to cope with the associated awkward calculations as she sought the non-integral charge per minute in Plumber's Fees and another did not recognise all the constraints in Question 9.

Table 3: Use of guess and check strategies

	9P			9N		
	Tries	Succ	Unsucc	Tries	Succ	Unsucc
Question 1 (Frank's)	6	5	0	9	9	0
Question 6 (Frank's)	3	3	0	1	0	0
Question 9 (Frank's)	4	3	1	1	1	0
Plumber's Fees	9	4	1	8	0	4*

The table gives the number of students from the group who attempted each question (Tries) and the number who used a guess and check strategy largely successfully (Succ) or unsuccessfully (Unsucc) for each question.
* Guess and check strategy prompted by interviewer when other methods did not seem promising.

Conclusion

The results shown above indicate several ways in which the students exposed to three years of teaching oriented towards problem solving have taken on some of the characteristics of able problem solvers. Forty years ago, Bloom and Broder (1950) described unsuccessful problem solvers as spending little time considering questions, but choosing answers on the basis of a few clues. The contrast between the initial behaviour of students from 9P and 9N on the Plumber's question illustrates this behaviour. The students with the "traditional" teaching grabbed at superficial relationships in the problem, manipulating the numbers to get a quick answer. Students from the class which emphasised problem solving were more prepared to explore the problems and were more organised, having come to use the tool skills (principally use of labelled tables) which have been modelled for them by their teacher. They rushed at answers less, they wrote more and they were more deliberate in their calculations.

Both groups of students spontaneously used a guess and check strategy when it was very clear how this should be done, but only students from the problem solving class used it successfully when it was not immediately obvious

what to guess and what constraints could be used to check it. The most important gain seemed to be their willingness to explore the problem, rather than try to manipulate numbers in the question to get a quick answer. Familiarity with guess and check has clearly strengthened the ability of students from the problem solving class to solve problems, and has substantially contributed to the resourcefulness that Frank seeks. However, their propensity to use this intuitive method is of concern. A better balance needs to be maintained so that the strengthening of the guess and check strategy does not inhibit their looking for the more powerful mathematical relationships that underlie problems.

References

Bloom, Benjamin S. and Broder, Lois (1950) Problem Solving Processes of College Students. Chicago: University of Chicago Press.

Charles, Randall. I. and Lester, Frank K. (1984) An Evaluation of A Process-Oriented Instructional Program in Mathematical Problem Solving in Grades 5 and 7. Journal for Research In Mathematics Education, 15 (1), pp15 - 34.

Groves, Susie & Kaye Stacey (1988) Curriculum Development in Problem Solving. In H. Burkhardt, S. Groves, A. Schoenfeld & K. Stacey (eds) Problem Solving: A World View, , Shell Centre For Mathematical Education, Nottingham, England, pp. 199 - 206.

Kantowski, M.G. (1977) Processes involved in mathematical problem solving. Journal of Research in Mathematics Education, 8 (3), pp 163 - 180.

Isaacs, Ian (1987) The effects of extended teaching in problem solving styles on the mathematical achievement of secondary school students in public examinations. In Ong Sit Tui (ed) Proceedings of the Fourth Southeast Asian Conference on Mathematical Education, Singapore: Institute of Education, pp 162 - 167.

Lester, Frank K. (1989) Reflections about Mathematical Problem Solving Research. In R.I. Charles and E. A. Silver (eds) The Teaching and Assessing of Mathematical Problem Solving, Laurence Erlbaum, U.S.A.

Stacey, Kaye (1989) Finding and Using Patterns in Linear Generalising Problems, Educational Studies in Mathematics, 20 (2), pp 147 - 164.

Stacey, Kaye & Susie Groves. (1985) Strategies For Problem Solving, Latitude Publications, Melbourne.

THE RELATIONSHIP BETWEEN MENTAL MODELS IN MATHEMATICS AND SCIENCE

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In the course of research now in progress, two externally similar but essentially different problems were presented to 7th to 12th grade students. For each of these problems students were asked to judge whether a process of successive division will come to an end: the mathematical problem referred to a line segment, the science one to a copper wire. Our data show that the external similarity of the two problems encourages younger students to use the same "finite" model for solving both problems whereas older students use the same "infinite" model when solving them. An intervention which suggested the appropriate responses was very effective only for the older students.

Research in science and mathematics education indicates that students tend to use inappropriate mental models for solving problems in these subjects (Fischbein, 1987; McCloskey, 1983). A central issue in mathematics and science education is that of the effects of various factors on student's response to a given problem. This raises more specific questions such as, How do factors related to the problem, i.e., its structure, the numerical data, the figural aspects and the content domain in which it is embedded, affect the student's solution? What effects do factors related to the solver, i.e., age, grade level and instruction, have on his/her solution to a given problem?

One way of investigating these questions is to present students with a variety of essentially similar problems and to examine the relationships between the specific features of each of these problems and students' responses (Silver, 1986; Stavy, 1990). A less conventional way, chosen in the present study, is to examine students' responses to problems which are essentially different (i.e., they stem from different theoretical frameworks and require different solutions) but are externally similar (i.e., they are figurally similar and refer to the same process).

In our study we chose two essentially different problems: A "mathematical" problem involving the potential infinity and a "scientific" problem involving the particulate nature of matter. Our main aims were (a) To determine whether students of different ages tend to erroneously produce the same response to both problems, or rather give differentiated, correct responses to each of them; and (b) To assess the effects of exposing students to the correct solutions on their responses to the problems posed.

Method

Subjects

Two-hundred upper middle-class students participated in this study. Fifty students each from the seventh, eighth, tenth and twelfth grade level were investigated. The tenth and twelfth grade students were studying mathematics as their major subject.

Seventh grade students had experience with elementary school geometry but did not yet receive any instruction in Euclidean geometry. At the time of the research, they had finished studying a chapter on the particulate nature of matter. Eighth grade students had been formally instructed concerning elements, compounds and the periodic table, while in mathematics, they had not yet been instructed in either geometry or infinite processes. Tenth grade students had studied basic Euclidean geometry (i.e., undefined and defined terms, axioms, postulates, definitions, theorems and proofs); they had not received any additional instruction concerning the structure of matter. Twelfth grade students took an introductory course in calculus in which they dealt with infinite series and limits. They also studied chemistry on a minor level (stoichiometry, the structure of the atom, etc.).

The Problems

1. Consider a line segment AB. Divide it into two equal parts. Divide each of these parts into two equal parts. Continue dividing in the same way. Will this process of division come to an end? Explain your answer.

2. Consider a piece of copper wire. Divide it into two equal parts. Divide each half into two equal parts. Continue dividing in the same way. Will this process of division come to an end? Explain your answer.

These two problems are fundamentally different. In the first an ideal, geometrical segment is being considered whereas the second involves a material wire. The adequate responses to these two problems are: In the case of the line segment the process of division is endless, whereas in the case of the wire the division process stops when reaching the atomic level (after which the elements considered lose their identity).

The Intervention

In order to help students confining their knowledge appropriately, they were exposed to the following query:

A student named Karen asked the following question:

> "I understand that when dividing a copper wire time and time again the process will end when reaching the atom level. In contrast, the successive division of a line segment is an endless process. Why is this so?"

Do you think that the above statement made by Karen is correct? Explain your answer.

Procedure

The two problems and the query task were administered to the students during one class period (about 45 minutes). In order to counter-balance the effect of the order of presentation of the two initial problems, half of the students in each grade level first received the mathematical problem while the other half first received the science one. Each problem appeared on a separate sheet of paper along with other, irrelevant questions and was taken away after the student had responded. Then the query task was administered.

Results

The Division Problems

Two types of responses were given to the two division problems: The process is endless; the process will come to an end (see Tables 1 and 2). In the case of the wire, as in the case of the segment, the frequency of "infinite" responses increased with grade level, while the "finite" responses decreased.

Explanations students gave to the two problems were essentially similar. Three main rationales were given to the "finite" answers: "We shall not be able to divide anymore because the segment/wire will become extremely small"; "The segment/wire is finite" or "There is a finite number of points/atoms in the segment/wire", and "We shall not be able to divide anymore as we shall reach a point/atom". Among the explanations given to the "infinite" answers, the most common one, at all grade levels was: "One can always divide by two". Other explanations were: "The segment/wire is infinite"; "There is an infinite number of points/atoms in a segment/wire"; "We shall reach a point/atom but it can also be divided."

Table 1: The Division of a Line Segment

	BEFORE INTERVENTION GRADE				AFTER INTERVENTION GRADE			
	7	8	10	12	7	8	10	12
The process is endless:								
Total:	26	50	78	78	28	54	80	86
- One can always divide by two	24	40	48	58	24	30	38	56
- The segment is infinite	-	4	26	20	4	8	24	20
- We shall reach a point, but it can also be divided	2	6	4	-	-	16	18	10
The process will come to an end:								
Total:	74	50	22	22	62	46	18	14
- The segment will become extremely small	44	18	6	4	14	30	8	6
- The segment is finite	8	22	2	6	6	8	4	2
- We shall reach a point	4	4	12	10	14	6	2	6
- We shall reach an atom	18	6	2	2	28	2	4	-
- I do not know	-	-	-	-	10	-	2	-

Table 2: The Division of a Copper Wire

	BEFORE INTERVENTION GRADE				AFTER INTERVENTION GRADE			
	7	8	10	12	7	8	10	12
The process will come to an end								
Total:	76	74	50	50	74	88	78	66
- The wire will become extremely small	36	28	22	26	26	28	22	30
- The wire is finite	10	6	2	-	8	6	10	12
- We shall reach an atom	30	38	26	24	40	54	46	24
- We shall reach a point	-	2	-	-	-	-	-	-
The process is endless:								
Total:	24	26	50	50	16	12	20	34
- One can always divide by two	20	22	42	40	10	6	8	20
- The wire is infinite	-	2	-	2	-	2	-	-
- We shall reach an atom, but it can also be divided	4	2	8	8	4	4	12	14
- I do not know	-	-	-	-	10	-	2	-

Response Patterns to the Problems

The four possible response patterns are presented in Table 3. Students who gave the same responses to both problems (finite or infinite) are included in the "concordant" patterns. Those who gave different answers to the problems are represented in the "discordant" patterns.

Table 3: Response Patterns to the Division Problems

Segment	Wire	Before Intervention Grade 7	8	10	12	After Intervention Grade 7	8	10	12
Discordant patterns:									
	Total:	14	42	32	44	12	54	60	56
Infinite	Finite	8	34	30	36	12	48	60	54
Finite	Infinite	6	8	2	8	-	6	-	2
Concordant patterns:									
	Total:	86	58	68	56	78	46	38	44
Finite	Finite	68	42	20	14	62	40	18	12
Infinite	Infinite	18	16	48	42	16	6	20	32
I do not know		-	-	-	-	10	-	2	-

The Discordant Patterns. Table 3 shows that the percentage of the students who showed discordant response patterns is relatively low. As can be expected, the most frequent one is the correct, infinite-finite response pattern. Yet, even among the higher grade levels it occurred in no more than 36% of the students. Very few students at each grade level gave "reversed answers" to these problems: a finite answer to the segment problem and an infinite answer to the copper wire one.

The Concordant Patterns. Table 3 shows that most students, at all grade levels, gave the same answer to both problems: either the finite-finite or the infinite-infinite. The finite pattern was frequent in the seventh grade and significantly decreased with grade level (X^2=34.76, df = 3, p< .0001). This decrease may indicate that students in the upper grade levels gave up their initial, correct finite response to the copper wire problem in favor of the infinite response. The infinite pattern was rare in the two lower grade levels and significantly increased with grade level (X^2 = 18.22, df = 3, p< .001). this increase is probably due to the extensive instruction of mathematics which apparently influenced the students' responses not only to the segment problem but also to that of the wire.

It is evident that almost all the students who showed a concordant response pattern also gave the same explanation to both problems. Some students explicitly referred to the apparent concordance of these problems. For instance, Dan , a tenth grader: "The division of the copper wire is endless and this is exactly as it was with the line segment".

Effects of the Intervention.

Table 1 shows that the effect of the intervention on the students' responses to the line segment problem was minor. It was noted mainly in the twelfth grade. About 8% of the students in this grade level, who initially argued that the process was finite, changed their response and correctly claimed that the process was infinite. They used a compromise notion of a divisible point namely, "We shall reach a point but a point can also be divided". Thus, it is evident that these students did not fully give up their initial belief about the particulate nature of a segment.

Table 2 shows that in the case of the copper wire problem, the intervention had a profound effect on students' responses: After the intervention the percentage of the finite, correct responses increased in grades 8, 10 and 12 while the percentage of infinite responses decreased at all grade levels. Thus, the minor intervention was enough to confine the use of the infinite model to the segment problem only.

After the intervention, less students in each grade level exhibited concordant response patterns while more students exhibited the discordant infinite-finite response pattern (see Table 3). The most pronounced tendency was a shift from the infinite-concordant pattern to the correct, differentiated infinite-finite one. The other concordant pattern, which consisted of finite responses to both problems, remained practically unaffected by the intervention while the discordant, finite-infinite reversed pattern almost disappeared.

It is noteworthy that many of the students who responded correctly to both problems after the intervention (but not before it) commented on the apparent differences between the two problems. For example, Rachel, a twelfth grader (adapting her answer to the line segment problem): "The copper is a concrete substance and a point is an imaginary creation."

Discussion

The two division problems compelled the students to deal with two entirely different entities: a line segment and a piece of copper wire. However, our data indicate that the majority of the students gave the same response to both problems.

Studies which examined differences in scientific problem solving between novice and expert solvers revealed that inexperienced solvers tend to approach a given problem according to surface features whereas experienced solvers refer to underlying scientific concepts and

principles (Chi, Fletovich, and Glaser, 1981; Larkin and Rainard, 1984). This may suggest that our subjects responded similarly to both problems because they formed mental representations which were based on the external figural similarity of the two entities (segment and wire) and the apparent identity of the process.

It was also found that the younger students, who had studied about the particulate nature of matter but not about line segments, tended to give finite responses to both questions. Similarly, the older students, who had been exposed to rather intensive mathematical instruction about geometrical concepts and infinite processes, tended to produce infinite responses. It is particularly interesting that students in the tenth and twelfth grades, who had, in fact, acquired the formal knowledge necessary for correctly solving each of these problems, had abandoned their finite, correct response to the copper wire problem in favor of the infinite one. This surprising result may stem from the fact that our older subjects are mathematics majors. It may very well be that should other students have been tested (for example, chemistry majors), they would have yielded other response patterns. This possibility is still under investigation.

Our results also indicate that a relatively minor intervention, which only suggested the two different, appropriate responses, was very effective for students in the higher grade levels who previously showed an infinite response pattern. It is most probable that this type of intervention has an impact only on students who hold the adequate, formal school-based knowledge in both science and mathematics to lean on when confronted with the correct responses. The intervention impelled the older students to disregard the external similarity and attracted their attention to the qualitative differences between the problems. Consequently, they were able to form different, adequate mental representations to each of them.

Implications

1. Teachers should be aware of their students' natural tendency to over-generalize from one domain to another, and attempt to help students define applications of newly acquired knowledge.

2. Teachers should assist students in forming mental representations that reflect the theoretical framework of the problem rather then its external features. More specifically, both the mathematics and the science teacher could relate to mathematics and scientific situations

which trigger the same response. Such situations can serve to open a debate on the scientific nature of the problems and their similar external features. This, consequently, may lead to a greater attention to the theoretical framework of posed problems.

3. For researchers who are studying the nature of students' responses in one specific content domain, it is imperative to consider the possible effects of knowledge acquired in related domains on students' responses.

References

Chi, M., Fletovich, P., & Glaser, R. (1981). Categorization and represent-ation of physics problems by experts and novices. Cognitive Science, 5, 121-152.

Fischbein, E. (1987). Intuition in science and mathematics: An educational approach. Holland: Reidal.

Larkin, J., & Rainard, B. (1984). A research methodology for studying how people think. Journal of Research in Science Teaching, 21, 235-254.

McCloskey, M. (1983). Naive theories of motion. In D. Gentner and A.L. Stevens (Eds.), Mental models. Hillsdale, New York: Erlbaum.

Silver, E. (1986). Using conceptual and procedural knowledge: A focus on relationship. In J. Hiebert (Ed.), Conceptual and procedural know-ledge: The case of mathematics (pp. 181-198). Hillsdale, N.J.: Erlbaum.

Stavy, R. (1990). Children's conceptions of changes in the state of matter: From liquid (or solid) to gas. Journal for Research in Science Teaching, 27, 147-266.

PUPILS AS EXPERT SYSTEM DEVELOPERS

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We report the results of an exploratory study which invited students to represent their knowledge about graph sketching in the form of an expert system, using a variant of the declarative programming language PROLOG. We point to some ways in which the construction of declarative rules provides students with new channels of communication about their graphical knowledge, and briefly consider the pedagogical implications of this approach.

Graph sketching is considered to be an important skill in mathematics not least because students who can produce accurate sketches can obtain information about the general properties of functions quickly. In this paper, we outline an exploratory study in which pupils developed their own expert systems in order to represent their knowledge and understanding of graph sketching (see Stevenson, 1990).

In common with all serious attempts to model student knowledge, we need to begin by asking how students represent their knowledge state. Accordingly, this paper is divided into two parts. Part 1 will briefly outline the results of a preliminary study which set out to map the ways in which students approach the problem of sketching graphs, and provide us with a framework for interpreting the students' programs. Part 2 will report the findings of the expert system study.

1. The graph-sketching study

The study aimed to consider two aspects of the process of graph-sketching:

a) What strategies do students adopt in sketching graphs?

b) What assumptions and expectations guide the decisions made by students about the final form of a sketch graph?

To address these issues, task-based interviews were undertaken with a group of students.

The students: Three students at various stages in their two year pre-university entry courses (known as 'A-level') volunteered to participate in the study. One student, aged 17, had completed one year of the course. A second student, also aged 17, had completed one year, taken the final exam and was proceeding to a more advanced course (known as 'Further Mathematics A-level'). The third student, aged 18, had completed two years of A-level courses in both Mathematics and Further Mathematics.

The interviews: The focus of the interviews was the process by which the student identified and communicated what he/she thought was important about a graph and the process of sketching. Each student was asked to sketch a number of Cartesian graphs. They were encouraged to indicate which regions of the graph they were looking at and speaking about. All the interviews were undertaken using paper and pencil. Student responses were taped and any written materials were kept for later analysis. Each interview lasted between one and one-and-a-half hours and was completed in one session.

Data analysis: The interviews were transcribed and then analysed using a classification which refers to the student's focus of attention to specific features

of the graph. Three types of feature were defined (see Table 1 below). Type one (F1) refer to *points*, that is features identified by the student which refer to specific points on the curve which are of interest in the sketching process. Type two (F2) refer to characteristics of the graph which are related to a subset of the domain. Type three (F3) refer to those properties of the graph which relate to the entire domain.

Type F1	Type F2	Type F3
1. Crossing the x axis.	1. Maxima,Minima,Inflexion	1. Graph transformations
2. Crossing the y axis.	2. Increases/decreases	2. Patterns,cycles
3. Stationary points.	3. Plateau regions	3. Symmetries
	4. Curves joining 1 or 3	4. Dispersion
	5. Steady change	5. Shape from area
	6. Non-steady change	6. Superposition
	7. Extra-/interpolation	
	8. Discontinuities.	
	9. Horizontal Asymptotes	
	10. Vertical Asymptotes	

Table 1: A classification of graphical features

The features F2 and F3 are derived from Janvier's extensive analysis of the interpretation of complex cartesian graphs (Janvier, 1978). He describes both kinds of features as "global" in that they are related to an interval or are "apprehended by more than a point to point process" (ibid 2.6). In our classification we designate as F2 those features which can be obtained by calculation over a subset of the graph's domain. F3 identifies those characteristics of a graph that are defined in relation to the entire domain of the function being plotted and which require the sort of extensive calculation that is inappropriate to sketching. F1 features do not form part of Janvier's original classification but are relevant to graph sketching.

Findings: Two basic strategies were found to be adopted by the students.

1. RECOGNITION STRATEGY. The sketcher recognises something about the sketch from the equation. This then guides the way the sketch is constructed and what information the sketcher needs to deduce from the equation. Two types of recognition are discernible: function and feature.

(a) *Function recognition...* What characterises this type of approach is that it is *visual* and *deductive*. The sketcher works from recognition of type F3 features and deduces the type F1 and F2 features. For example, one student was able to produce an accurate sketch of $1/(x^2-1)$ entirely by transformations of the graph of x^2.

(b) *Feature recognition...* The sketcher identifies a feature (usually F2) of the graph from the equation. This enables him/her to build up the sketch on the basis of type F1 and F2 features. Another student, in sketching the same graph. immediately recognised the vertical asymptotes at $x=1$ and $x=-1$ and the possible symmetry implied by the squared term.

2. NON-RECOGNITION STRATEGY. The sketcher collects information about type F1 and F2 features in a systematic way and searches for clues as to the shape of the graph. The process is complex and appears to rely upon the visual

coding of images in the memory. As one student put it "I haven't got much idea what this one looks like, but by doing a few points I probably will. It will trigger something off in the memory".

Pupils' interpretative background for graph sketching seems to be shaped by a process which generally develops according to the following pattern Knowledge and extensive practice at using a taught method build up experience Expertise grows over a period of time as students are exposed to a variety of functions and also have to study some in great detail. The sort of knowledge that is acquired in this way is direct, unreflective and 'self-evidently true' students referred to it as "instinctive" — they are able just to "join up the points" without deliberation. Hence, students know what to do, although they may not be able to explain why they do it. From this background come expectations and assumptions about curves — for example, they are smooth and continuous, although what this means is not precisely stated. Similarly, cartesian graphs are relatively simple in the sense that they contain a small number features which recur regularly.

Graphs are forms of communication. To sketch graphs effectively implies a facility with a background of shared meaning which enable the "code" of graphing to operate. Making sense of a graph must be relative, therefore, to the framework of rules and activities that constitute the *language-game* of graphs in the Wittgensteinian sense of a partial system of communication and activity. Any attempt to investigate this interpretative background of students must try to encourage them to make explicit that which is implicit in their actions. We suggest that expert systems may provide one way of doing this.

2. The Expert System Study

Expert systems are computer programmes that perform some of the tasks of human experts, e.g. they can give advice, diagnose or find faults. Knowledge is presented in rule form which can be used by the computer to respond to questions as an expert might. Expert systems can also give an account of how they have used rules to come to the conclusions presented to the user (Cotterell *et al.*, 1988).

A commonly-used programming language for implementing expert systems is PROLOG (PROgramming in LOGic). This is a *declarative* language (unlike, for example, Logo — which is *procedural*) which uses descriptions of some problem in terms of facts and rules to deduce solutions. This makes for potentially interesting mathematical expression, in that a program is essentially nothing more than a data-base which consists of knowledge elements which the learner 'knows'.

In the literature, expert systems are normally created *by* experts *for* users. Our intention is to reverse this process: we wish to explore what happens when pupils express their own knowledge-state in the form of a PROLOG program. For us, the important feature of expert systems is the need for those who create them to make *explicit* the knowledge that is to be used. Experts draw upon knowledge in a specific domain that consists of facts, procedures and rules of thumb, built up over a period of time. *Creating* an expert system requires the expert to reflect on his/her performance, to identify and formalize what is used to deal with particular problems. In the case of graph-sketching this means

making explicit not only the calculation of points, but also the sorts of judgements that are made in completing a sketch. For example, how are points to be joined up? What guides the choice of shape? To answer such questions, the expert must stand back and try to build a comprehensive representation of knowledge that can be used by the non-expert.

Lippert (1987) lists a number of claimed benefits for this kind of activity which include the development of critical thinking; decision making; organisation and communication of knowledge; development of qualitative and conditional reasoning. Cotterall *et al.* (1988) identify five uses for expert systems in schools: as a resource; as a simulator; as a tool to improve effectiveness in students and teachers; as an intelligent tutor and as a means of exploring knowledge. In mathematics education, PROLOG has been used as a theorem-checker (Holland 1985).

Unfortunately, the syntax of PROLOG has won a deserved reputation for its obscurity and unfriendliness: this has given rise to a number of 'front-ends' which have been developed for use in schools. These have attempted — with varying degrees of success — to shield the learner from the more baroque elements of raw PROLOG. One of the challenges for constructing such a front-end, is to develop an interface which enables learners to deal with the syntax *without* adding to the obscurity of the language itself: in other words, to develop a means of *access* to PROLOG rather than a *barrier* to it. One reasonably successful attempt has been MITSI[1] which allows the learner to deal with both qualitative and quantitative facts and has a simple dialogue facility that enables it to remember and prompt the learner with facts that it has been told.

A typical MITSI fact has the form: `<name> relation <name>` (e.g. `the-cat sits-on the-mat`). A rule has the form: `A if B and C and D`.... For example,

```
the-cat is content
      if the-cat sits-on the-mat and
      the-cat has milk and
      the-cat makes purring-noises
```

In answer to the MITSI question: `the-cat is content?` PROLOG will search its collection of facts to check whether it can confirm each of the antecedents. If it can,it will reply `yes`. If it does *not* have the facts, MITSI will ask, for example:

```
Is it true that the-cat sits-on the-mat?
```

It will explain its chain of deduction if asked `why?`, after a successful query. Using MITSI without rules enables it to be used as an expert system, since it will ask the user for any facts it does not have. Variables are introduced easily by using the underscore (_) as a prefix for any text string. Thus the rule above becomes:

```
_the-cat is content
      if _the-cat sits-on  the-mat and
      _the-cat has milk and
      _the-cat makes  purring-noises
```

[1] Man- (sic) In-The-Street Interface.

Variables can also be introduced for each of the other names in the clauses. Thus `_the-cat sits-on _the-mat` will be compared with any fact having the relation `sits-on` when MITSI checks to see if it can confirm the rule. Numerical calculations and list processing can also be carried out using MITSI.

The students. The task that the students were given was to build an expert system in MITSI which could give advice on graph sketching to other students. The participants were volunteers, taken from the first and second year of A-level mathematics. Two mixed-sex pairs actually performed the task, one from the first year and one from the second year. Pair A (both aged 17) had completed A-level Mathematics in one year and were proposing to take Further Mathematics in the second year. Pair B (both aged 18) had taken A-level Mathematics and Further Mathematics concurrently: hence both pairs had experience of graph sketching. The boys in each pair had considerable computing experience and expertise, the girls had very little. PROLOG was new to all students and so, in preparation for using MITSI in a complex task, the students spent about 10 hours prior to the task learning the language. Each pair spent about five hours on the task in one continuous session.

The task. The session was divided into two parts: a role play and the main task. The role play was intended to simulate the sort of interaction a person might have with an expert system, and the students played at questioning and being questioned by an expert. In the second part, the students, working in pairs, were given an open-ended task: to produce an expert system on graph sketching. No other information or guidance was given. Student interactions were taped and later transcribed.

Findings. We begin with the MITSI rules developed by Pair B at their first attempt. Table 2 provides their knowledge base with the left and right columns providing a linguistic and mathematical formulation of the MITSI rules in the centre.

In the first place, it is instructive that the rules are essentially concerned with types F1 and F2 features of graphs (in particular, F1.1, F1.2, F1.3 and F2.1). Pair B have essentially translated their graph-sketching behaviour into the new medium — MITSI. They have undertaken a symbolic translation, but at this stage have not reconstituted their knowledge state using the new codes available. Second, Pair B have reflected on their graph-sketching behaviour *explicitly*. This is no mean feat, and constitutes a meta-cognitive shift which, we argue, is an essential corollary of developing a declarative description of the kind we are considering.

Linguistic	MITSI	Mathematical
$y=ax^3+bx^2+cx+d$ has turning points if the discriminant of $dy/dx = 0$ is greater than or equal to zero	(_a _b _c _d) has turnpts if _p = ((_b* _b)-(3* _a* _c)) and 0 LTE _p	$y=ax^3+bx^2+cx+d$ has turning points if $dy/dx= 3ax^2+2bx+d=0$ and b^2-3ac 0
$y=ax^3+bx^2+cx+d$ has one turning point if the discriminant of $dy/dx = 0$ is equal to zero	(_a _b _c _d) has oneturnpt if _p = ((_b* _b)-(3* _a* _c)) and _p equals 0	$y=ax^3+bx^2+cx+d$ has one turning point if $dy/dx= 3ax^2+2bx+d=0$ and $b^2-3ac =0$
$y=ax^3+bx^2+cx+d$ has two turning points if the discriminant of $dy/dx=0$ is greater than zero	(_a _b _c _d) has twoturnpts if _p = ((_b* _b)-(3* _a* _c)) and 0 less _p	$y=ax^3+bx^2+cx+d$ has two turning points if $dy/dx = 3ax^2+2bx+c = 0$ and $b^2-3ac > 0$
$y=ax^3+bx^2+cx+d$ turns at the points x1 and x2 if y has turning points and x1 and x2 are the roots of $dy/dx=0$	(_a _b _c _d) turns-at (_x1 _x2) if _q = ((-1 * _b)) and _p=(SQRT(_b* _b-3* _a* _c)) and _x1 = (_q+_p)/(3* _a) and _x2 = (_q-_p)/(3* _a)	$y=ax^3+bx^2+cx+d$ has turning points at x1 and x2 if y has turning points and x1 = -b+ (b²-3ac)/3a and x2 = -b- (b²-3ac)/3a
$y=ax^3+bx^2+cx+d$ has a maximum point if its second derivative is negative	(_a _b _c _d _x) is max if _p = (3* _a* _x* _b) and _p less 0	$y=ax^3+bx^2+cx+d$ has a max at x1 if $d^2y/dx^2 <0$ at x1
$y=ax^3+bx^2+cx+d$ has a minimum point if its second derivative is positive.	(_a _b _c _d _x) is min if _p = (3* _a* _x* _b) and 0 less _p	$y=ax^3+bx^2+cx+d$ has a min at x1 if $d^2y/dx^2 >0$ at x1

Table 2: MITSI rules with linguistic and mathematical formulations

Third, we notice that pair B have avoided the *global* features of graphs which characterise F3 strategies. Before we consider the implications of this, we move on to discuss the next step in the task. This involved discussing (with each other and with the researcher) ways of formulating rules which focussed on transformations and features of the whole graph. The students then returned to the computer and evolved a program to represent facts of this kind. Pair A worked with quadratic functions and produced the following (the classification shape/transformation is ours not the students').

Shape

```
1. squared   gives u-shape[2]
2. squared gives even
3. graph-of-x   gives straight-line
```

Transformations of graphs

```
4. negative gives   reflection-in-x-axis
```

[2] MITSI rules are not numbered: we have numbered them for reference.

```
5. add-constant  gives shifted-up-by-constant
6. function-of-x-plus-constant gives shifted-left-by-constant
7. squared  gives minimum-turning-point-at-x-equals-zero
8. negative gives all-maxima-turn-into-minima
9. negative gives all-minima-turn-into-maxima
10. squared  gives root-at-x-equals-zero
11. function-of-x-multiplied-by-constant  gives squashed-up-
        by-a-factor-of-constant.
12. multiplied-by-constant  gives height-increased-by-a-
        factor-of-constant
13. reciprocal-of-function-of-x  gives horizontal-asymptote-y-
        equals-zero
```

If we compare these formulations to those in Table 2, it is noticeable that a number relate transformations of graphs with transformations of the associated algebraic expression. So, for example, in fact 5, f(x) is transformed to f(x)+c; the fact describes what would happen to the graph — it is translated parallel to the y axis. Fact 6 states that if f(x) —> f(x+c) then the graph is translated parallel to the x axis. It seems that the formulations are being *re-coded;* there is little sense in which these facts symbolically *restate* what graph-sketchers routinely *do*.

Pair A (and Pair B) found the task of describing type F3 features difficult. This difficulty is reflected by the structure of pair A's response. It begins with short and unelaborated statements about shape (rules 1-3), and moves to longer and more detailed statements about things that they can describe: transformations. One possibility for their difficulty is that type F3 features are essentially ***visual***. They provide the framework which bind the F1 and F2 features into a coherent perceptual object. F3 features can be "shown but not said". The text-based nature of PROLOG requires the students to translate from visual to verbal forms of expression *without* any available non-verbal representation on the screen[3].

Pair B developed rules for cubics because they felt that they knew about this group of functions and chose to restrict their initial attempts to something manageable. The facts below relate type F2 features such as maxima and minima to type F1:

```
1. point-of-inflexion-may  give three-equal-roots
2. point-of-inflexion-may  give only-one-real-root
3. cubic-equations-without-statpts  give point-of-inflexion
4. cubic-equations  give continuous-curves
5. stationary-points-of-same-signs  give only-one-real-root
6. stationary-points-of-different-signs  give three-real-
        distinct-roots
```

Here fact 4 is the only general statement about cubics The other facts are derived from things they had been taught or could be obtained from a text book. It is interesting to note the tentative nature of rules 1 and 2, which may indicate that the students were aware of the possible complexities involved and that sketching is often — *in practice* — not an exact science: there is room for judgements and

3 Since this study, the version of PROLOG used has acquired graphics.

hunches (which of course, are hard to incorporate into an elementary expert system!).

Some tentative conclusions

This exploratory study has raised a number of interesting issues. Expert systems can provide the student with a way of representing and testing what he/she knows (or may not know). Requiring both rigour and precision, the formulation of facts and rules is a process which helps the student to clarify and elaborate his/her thinking. On the negative side, we are a long way from having available media through which students might express their knowledge as an expert system *without* being constrained by the generally stilted and unforgiving syntax for which PROLOG is (in)famous. We also point to a more general issue, which concerns the extent to which mathematical learning (or indeed mathematical knowledge) is helpfully represented as a set of declarative 'rules' at all.

We suggest that expert systems have a role in helping students to be aware of their knowledge state in two ways. In the case of recognition strategies it can help students to formulate the graphical transformations used to produce a sketch, as outlined in Pair A's facts. It can also help students to collect rules of thumb which guide their search for overall shape — as with Pair B. As for non-recognition strategies, Pair B's collection of rules shows how the the standard techniques used for finding information about F1 and F2 features can be implemented in a declarative environment.

We end with a speculation. Elsewhere, we have pointed to the distinction between local and global mathematical understandings (see Hoyles & Noss, 1990), and argued that global understandings (GMU) within mathematical discourse *necessitate* pedagogical intervention at some level. We suggested that the computer has a critical role as a mediator in the intervention process, and is governed by the need to formalize knowledge, which in turn provides pupils with scaffolding to make sense of the (global) mathematical ideas underlying their activities. These assertions flowed from work over a considerable period with Logo, a procedural language. We would like to know what possibilities are further opened up by the construction of programs by students based on declarative representations of what they know, and how they know it.

REFERENCES

COTTERALL A.,ENNALS R.& BRIGGS J.H. (1988) *Advanced Information Technology in Education and Training*. London:Edward Arnold

HOYLES C. & NOSS R. (1990) Deconstructing Microworlds. In: FERGUSON D. (ed.) *Advanced Technologies in the Teaching of Mathematics and Science*. Berlin: Springer-Verlag.

HOLLAND G.(1985) Computer Modelling of Mathematical Problem-Solving Processes as an Instrument of Competence Analysis. *PME IX*. 24-29

JANVIER C. (1978) *The Interpretation of Complex Cartesian Graphs* .Sheffield:Shell Centre

LIPPERT R.(1987) Development of Expert Systems:An Instructional Strategy for Dealing with Misconceptions in *Proc. 2nd Int. Seminar on Misconceptions and Educational Strategies in Mathematics and Science*, 275-288.

STEVENSON I. (1990) *Expertise Observed:A Study of Sixth Formers Sketching Graphs.* Unpublished MA Dissertation.University of London.

DRAWING - COMPUTER MODEL - FIGURE

Case Studies in Students' Use of Geometry-Software

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Summary: Usually, use of geometry software on computers is taken as a new tool to produce drawings more easily (or in a more difficult way). The "geometry-computer" seems to bring in no fundamental change in doing/learning/teaching geometry. By presenting three cases (an analysis of students' constructions task, students' use of an ITS-prototype and an "intelligent" geometry-inference feature), the paper gives reasons to deny this assumption and offers a more complicated, more appropriate view on students' use of geometry-software.

1 Drawing versus Figure: A Fundamental Complementarity in Geometry

Using the "triangle" pertinent in Mathematics Education of symbol/ sign-object-concept (cf. [Steinbring] p.29f; also [Ogden and Richards]), Geometry can be characterised by the use of different systems of signs (as "drawings", verbal, sometimes standardised descriptions of configurations) to describe concepts (which are "figures" representing logical relations and 2- or 3-dimensional configurations; for the distinction of "drawing" and "figure" cf. [Laborde1989] and [Parzys]). For certain uses like technical drawing, geometrical constructions, views and perspectives (the "drawings" as the material representation of the figure) are highly standardised in order to secure the relation between drawing and figure. Geometry heavily plays on the interaction of drawing and figure using visual perception as agent between signs and concepts. At a first sight, the use of computers and geometry software, the "geometry-computer" (the system of hardware and geometry-oriented software) brings in no fundamental change in doing/learning/teaching geometry. It is just a different way, a new tool to produce drawings which may be easier (or more difficult) to handle. The paper will show reasons to deny this assumption and offers a more complicated, but more appropriate view on this question.

2 Students' Use of Geometry-Software

2.1 Drawing a Square: The Seduction of Perception

In February 1989, 12- to 14-year old students of a French "collège" (i.e. 7th-8th grade) were given the task to construct a square using "Cabri-géomètre" (for details on "Cabri" cf. [Baulac et al.], for a

detailed report on the experiment [Sträßer]). The students were offered the following menu: They could "create" points, line-segments, lines defined by 2 points and circles defined by center and radial point and could "construct" points on an object, the intersection of two objects, the midpoint of a segment, the perpendicular bisector, lines parallel to given lines/segments, perpendicular lines and symmetrical points (using point and line symmetry). The students started from the given line-segment AB with the square to be constructed in a way that its sides are of the length of AB. When they said to have finished the construction, they evaluated the construction by "dragging" one vertice of the quadrilateral to see if the figure was a square "in the sense of Cabri" (if it remains a square while dragging one of its vertices). After a correct solution, they were asked to find a different solution whith the sides having arbitrary length. The constructions of the students were protocolled by means of the protocol-function of "Cabri".

Analysing the task, it is clear that the construction of the square needs two fundamental concepts: The right angle concept, which is a usual concept for students of that age and can be used directly in "Cabri" by means of "perpendicular line" in the "construction"-menu The second concept to be used is the equidistance property of the points of a circle to the centre. Indeed it has been shown that if children are very early able to recognize a circular shape, they only later conceptualise a circle as a set of equidistant points (cf. [Artigue&Robinet]). It is important to note that even if grade 6 students know how to copy a given length using compasses, they do not perceive the point D as an intersecting point of the circle of centre C and the straight line Δ' (see fig.1). For the students, the use of compasses is similar to the use of a graduated ruler.

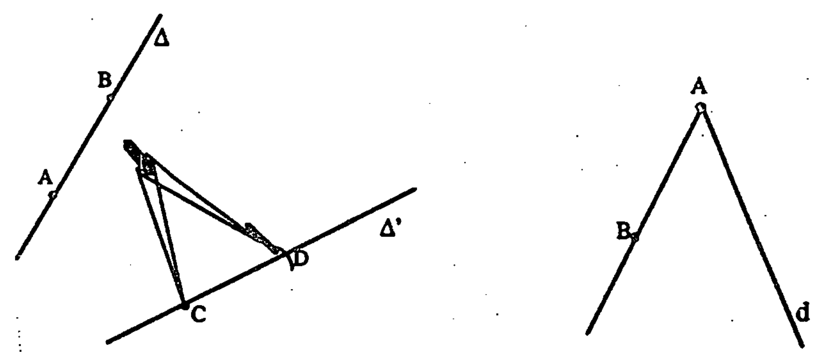

Figure 1: Using Compasses for Copying a Segment-line

Since "Cabri" does not allow to directly obtain segments of given length, the user can (in a configuration like the right part of figure 1) construct on d a segment AC equal to AB, by means of the item "Circle def. by center and radial point" (or other geometric properties like symmetry)..By dragging, "Cabri" allows the rejection of a construction by the eye and thus gives a meaning to the characteristic property of a circle. The "milieu" (that is the antagonistic system to the learner [Brousseau]) is organized in a way, which gives a specific rôle to perception [Laborde1989]). In a situation like this, perception can be used as an instrument of validation, but not as an instrument of solution.

The task of drawing a square was given to four pairs of students and two students using the computer alone[1].. About half of the students were familiar with the Macintosh-environment in general, none of them had special experience with "Cabri" The students were introduced to "Cabri" and the menu-items offered and an example of "dragging" was shown. Before they started the construction, the interviewer/ supervisor asked them to recall the defining properties of a square. All of them first stated the "equal length" property (equilateracy), while the orthogonality of the sides sometimes was only mentioned after being confronted with the example of a rhombus.

For a complete picture of the students' constructions we send the reader to [Laborde&Sträßer], pp. 174f and concentrate here on special cases: Three of the students gave solutions using perpendiculars and parallels to secure the right angles and a circle/circles to secure the equal length property. To give an idea of this type of solution, figure 2 shows the first construction of a student (including the automatically generated verbal description of the construction). From the description, it should be clear that the quadrilateral will remain a square even when being dragged - and the student (and the interviewer) was happy with this solution. There were also more complicated "correct" solutions using e.g. specific properties of the square (the centre of the inscribed circle is the intersection of the perpendicular bisectors of the sides of a square, etc.) - especially with no length of the side being fixed.

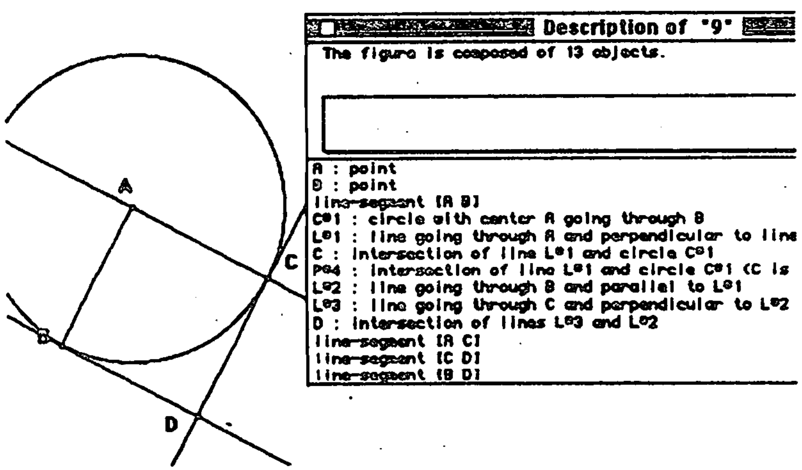

Figure 2: Construction of Student n° 6

[1]The paper speaks of "student"-solutions also when analysing the solution of two students.

A different type of "**solution**" is to be discussed here too: In their first construction, three of the students secured the orthogonality by means of perpendiculars (and parallels) while securing the equilateracy estimating by the eye and/or numerically measuring the length of the sides (in the version of "Cabri" used: up to millimetres) and adjusting them to equal value. One student measured the distance of a parallel to AB from the segment by means of this menu-item, placed a third point on this parallel and then measured the distance from that point to the fourth point of the "square" by the same procedure. (cf. figure 3 for the completed construction). The student was not too happy with this solution because from the measurement, one can see that one side of the quadrilateral has the wrong length.

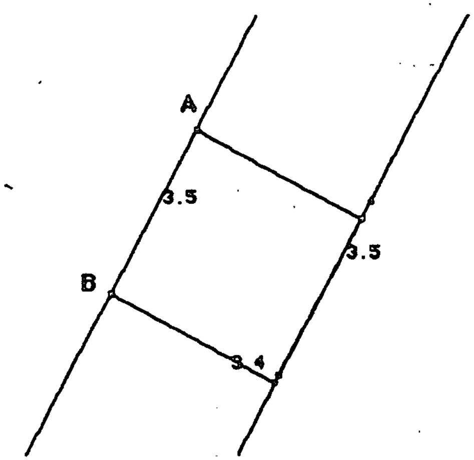

Figure 3: Construction of Student n° 5

An overall evaluation of the task shows that students had no problems with the orthogonality - last not least because a simple way to secure orthogonality was using the appropriate menu-item. Additionally, orthogonality can easily be controlled by perception - at least roughly. This may be taken as the indication, that the "drawing" aspect of a geometry-computer can easily be handled even by unexperienced users. For the equilateracy of the sides of the square, there were three possibilities (judgement by the eye, measuring and circle constructions) and all possibilities were actually used. The case study shows a special attraction of a mere perceptive approach with the construction, a mere "drawing"-approach. With this geometry-computer, a specific procedure (in this case: "dragging") can be additionally used to distinguish between the "drawing"- and the "figure"-aspect of the students' constructions.

2.2 Tutorial Help: Fallacies of an Intelligent Tutoring System.

Using a prototype of an intelligent tutorial system we analysed students' use of this specific "geometry-computer" integrating tutorial help facilities. The prototype uses capacities of "Cabri-géomètre" and was built to study conditions of realisation of an interaction between "Cabri" and "Hypercard". A Hypercard's stack presents the task (of drawing a square) to the student and additionally offers an evaluation of the construction in "Cabri" and some appropriate help. The student's construction is evaluated by the prototype, which consequently chooses helps from a variety of help-cards which are offered to the student[2]. Following the analysis of the knowledge necessary for the construction (cf. part 2.1 of this paper), the protoype offers two kinds of help:
• help about angles which verbally links right angles and perpendicular lines and gives hints for their use in "Cabri".
• a more elaborated help for the transfer of length which presents a new task to the student. The new task should be more accessible to students in difficulty with the length transfer. The help is constructed on the basis of the task analysis in part 2.1 of the paper and should help the student with the solution of a new task instead of immediately bringing some indications on solution procedures for the square construction. The solution should be transferable to the square construction.

With this prototype, a student , alone with computer, is guided during a task of construction. The prototype was tested with 13 year old students in a French college near Grenoble. After an introduction to the use of "Cabri", the students were alone with the computer. We collected a set of protocols which permit a first analysis of the use of the prototype (for details cf. [Laborde& Sträßer]).

Two main difficulties were observed:
• The first one is a consequence of the evaluation realised in the prototype. The evaluation starts with the number of explicitly constructed line-segments. A student, who creates perpendicular-lines and does not explicitly construct each side of the square as a line-segment, receives a message which only mentiones the number of line-segments. Nevertheless, this student may have solved the problem of equilateracy and should be given a more appropriate evaluation. This

[2]The transfer of information from "Cabri-géomètre" to "Hypercard" was a major problem in the development of the prototype. The originality of the interaction consists in the nature of collected information. In fact, evaluation not only concerns the final product, but also elements of the process of construction. At any time of the interaction, the students can go back to his/her construction. The construction is always avalaible and can be modified.

deficiency of the evaluation has prevented most of the students to use helps available in the prototype.
• A second difficulty comes from insufficient mastership of the software "Cabri" by many students. To give an example: the students try to refer to an intersection of lines visible on the screen but not (yet) explicitly constructed by them[3].

Apart from deficiencies of the prototype, the experimentation clearly shows the interaction between the modelling of the student's ideas (the "figur") and the computer-model within the prototype. Consequences from the difference between the knowledge representation by the students and the computer led to the problems in the experimentation, but can be interpreted in a more substantial way: As long as there is a gap between these two representations, it is difficult to count on an automatised evaluation of students' geometric constructions. A solution to this dilemma (namely totally giving the control over the evaluation to the user, but developing an "intelligent" geometry-computer by offering an inference mechanism) will be analysed in the next part of the paper.

2.3 "Intelligent" Software: Problems and Potentials

Besides the development of a geometry-ITS, there is a different way to play on the relation of drawing and figure by using a geometry-computer: At least in the version 2.x on Apple-Macintosh-machines, "Cabri-géomètre" offers the possibility of evaluating certain geometric properties of a configuration. The user can identify elements in a drawing and question the geometry-computer whether they fulfil relations like collinearity of three points, belonging of a point to a segment, line or circle, parallelism or orthogonality of segments/lines and congruency of segments. With this potential, the geometry-computer is more than a tool to (easily or painfully) produce drawings, but behaves as if it can "prove" geometric properties.

To give an example of this feature, "Cabri" will identify the segments joining the two points at the intersection of a circle and its diameter and a third point on the circumference of the circle as orthogonal (property of Thales). Dragging the third point on the circumference or changing the diameter of the circle will not affect the reaction of the geometry-computer. Is this a "proof" of the "Thales"-property ?

[3]The two difficulties (especially the first one) made impossible a detailed analysis of the potentials of the prototype. Because of this difficulty, we do not know if the "length help" offered really permits the discovery and subsequent use of "length transfer" necessary for the square construction. As a consequence, a second prototype is actually built with the same intentions - preserving the basic elements of the first one (particularly the didactic analysis). The most important modifications concern the evaluation of the the student's construction. In particular, the second prototype uses an analysis based on intentional elements of student's procedure.

A didactical evaluation of this feature again partly depends on the internal representation of the geometry-computer and the way, the machine produces its reaction. From a traditional mathematical standpoint, a numerical calculation of specific values of a drawing cannot be taken as a proof. A didactician could search for configurations producing answers not adequate theoretically, mathematically - for "monsters" in computer-geometry. The monsters can show the potential and limitation of the automatic evaluation and advance didactical theorising of computer-based geometry.

Example:
With two circles K_1 and K_2 of equal diameter and a tangent T to K_1 parallel to the segment joining the two midpoints M_1 and M_2, euclidean geometry secures that T is also a tangent to K_2. In version 2.0 of "Cabri", the intersection of T and K_2 produces two points (contradicting the euclidean view and showing some deficiency in the internal computer-model of "Cabri"). With a description of this construction sent to the program-developers, they identified a "bug" in the program (namely in the calculation of the intersection) and corrected the program. More recent versions now produce only one intersection - as is desirable from the euclidean geometry[4].

What is important with this example is the indication that the computer-model of the configuration may be in conflict with the configuration "prescribed" by (Euclidean, theoretical) geometry as well as the idea, the "figure" of the user. As long as the internal representation of the geometry-computer differs from the mathematical model of the configuration, a didactical analysis of computer use in geometry teaching/learning has to take into account this difference[5].

3 Conclusion on the Role of the Computer Model: Between Drawing and Figure

Looking back at the three "cases" presented in part 2, we can learn about the interaction of a geometry-computer with a learner/user in geometry: The first case clearly shows, that the "drawing" aspect of a geometry-computer can easily be handled even by unexperienced users. There is a special attraction of a mere perceptive approach in a computer-based construction. Geometry-computers try to counterbalance this bias by specific procedures ("Cabri" and "Geometers' Sketchpad" with "dragging" and other mechanisms, the "Geometric Supposer" by presenting different drawings of the same

[4]A German software named "GEOLOG" produces an empty intersection. "Geometer's Sketchpad" identifies no or two points at the intersection, depending on the position of the respective elements of the drawing.

[5]In this paper, we do not speculate on the possibility of definitely closing the gap between the theoretical and the computer modelling of geometric configurations - but start from the assumption, that the gap can definitely not be closed.

figur, "GEOLOG" by offering a discontineous "dragging" option etc.). All these procedures can be understood as a way to distinguish between the "drawing"- and the "figure"-aspect of the students' construction. The ITS-prototype in part 2.2 in principle offers an computer-based internal evaluation of the drawing by modelling the "figure"-aspect of the construction. Even if we overlook deficiencies of this prototype, the experimentation clearly shows the interaction between the specific modelling of the student-computer-interaction within the prototype - and the consequences from the difference between the knowledge representation by the student and the computer. What is important with the example presented in part 2.3 is the conflict between the computer-model of the configuration and the configuration "prescribed" by (Euclidean, theoretical) geometry.

As long as the internal representation of the geometry-computer, the computer-model, differs from the mathematical model of the configuration (cf. part 2.3), as long as it differs from the ideas of the user (cf. parts 2.1 and 2.2), a didactical analysis of computer use in geometry teaching/learning has to take into account this difference. The cases presented can be taken as good indications that the differences are not accidental (in terms of specific features of the software used), but should lead to a refined model of computer-use in geometry with the components "drawing - computer model - figure" and additional attention to the social "meaning" of geometry and its use.

References

Artigue M., Robinet J., (1982) : Conceptions du cercle chez les enfants de l'école élémentaire Recherche en didactique des mathématiques 3 (1), 5-64.

Baulac, Y., Franck - Laborde, Jean-Marie. (1988). Cabri-géomètre, un logiciel d'aide à l'apprentissage de la géométrie (logiciel et manuel d'utilisation) . Paris: Cedic-Nathan.

Brousseau, G. (1988). Le contrat didactique: le milieu. Recherches en Didactique des Mathématiques, 9(3), 309-336.

Laborde, C. (1989). L'enseignement de la géométrie en tant que terrain d'exploration de phénomènes didactiques. Recherches en Didactiques des Mathématiques, 9(3), 337-364.

Laborde, J.-M. ;. Sträßer, R. (1990). Cabri-géomètre: A microworld of geometry for discovery learning. Zentralblatt für Didaktik der Mathematik, 22(5), 171-177.

Ogden, C. K. ;. Richards., I.A. (1923 /1974). The Meaning of Meaning (deutsch: Die Bedeutung der Bedeutung); (1. deutsche Aufl.: 1974, nach der 10. engl. Aufl. von 1949). London: Routledge and Kegan Paul (deutsche Ausgabe: Frankfurt/M.: Suhrkamp).

Parzys, B. (1988). Knowing vs. Seeing. Problems of the plane representation of space geometry figures. Educational Studies in Mathematics, 19(1), 79-92.

Steinbring, H. (1989). Routine and Meaning in the Mathematics Classroom. For the Learning of Mathematics, 9(1), 24-33.

Sträßer, R. (1989). Students' Constructions and Proofs in a Computer Environment - Problems and Potentials of a Modelling Experience. Paper presented at the Workshop on "Modelling the Student Knowledge - the Case of Geometry" (Grenoble, November 13-16, 1989). Grenoble.

OVERCOMING OVERGENERALIZATIONS: THE CASE OF COMMUTATIVITY AND ASSOCIATIVITY

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Abstract

This study was aimed at (a) Examining the extent to which prospective teachers tend to perceive the commutative and the associative laws as logically dependent, and (b) analyzing the effects of exposing prospective teachers to examples of operations that obey only one of these laws on their perceptions of the logical relationship between the two laws. The Results indicate that a substantial number of preservice teachers (61%) believe that all binary operations are either both commutative and associative or neither commutative nor associative. A short and quite simple intervention was very effective in raising the prospective teachers' awareness of the logical independence of the two laws.

In recent years much has been written about children's and adults' tendency to overgeneralize patterns they had observed in relatively limited number systems (Bell, Fischbein, and Greer, 1984; Hart, 1982). Studies aimed at helping students overcome overgeneralizations have shown that confronting students, prospective teachers and teachers with counter examples of their over-extended rules is quite effective in bringing them to a realization that their conceptions are inadequate (Swan, 1983; Greer, 1987; Tirosh and Graeber, 1989). The present study deals with an overgeneralization related to the associative and the commutative laws.

The commutative and associative laws are, traditionally, first introduced in elementary school with reference to the basic mathematical operations: addition, subtraction, multiplication and division. Each of these four operations is either both Commutative and Associative (C&A) or Neither Commutative Nor Associative (NC&NA). It

is, therefore, very likely that students will get the wrong impression that all mathematics operations are either C&A or NC&NA. Based upon this supposition, Hadar and Hadass (1981) offer a set of counter examples which consists of operations that obey one of these two laws but not the other. They suggest that a careful examination of such examples may encourage students to refute the above mentioned improper generalization and realize that the two laws are, in fact, logically independent (Ibid, p. 535).

In a previous phase of this study it was observed that all the involved, prospective elementary teachers, perceived the commutative and the associative laws as logically dependent (Tirosh, Hadass, & Movshovitz-Hadar, 1990). As teacher educators, we were interested in (a) assessing whether prospective junior high school teachers, who specialized in mathematics, also believe that associativity and commutativity imply each other; and (b) examining the effects of exposing this population to some examples of operations that are Commutative and Not Associative (C&NA) or operations that are Not Commutative but Associative (NC&A) on this belief.

Method

Subjects

Eighteen prospective teachers at a teachers college in Tel-Aviv participated in this phase of the study. All were females in their third year towards a B.Ed. degree in mathematics education for junior high school. These prospective teachers had a rather strong mathematical background including Cantorian Set Theory and Group Theory.

Pre-assessment instrument

The following five-item questionnaire was administered after a brief, formal review on the notion of binary operation and of the commutative and associative laws;

1. Is there any operation defined on a set S, that is both commutative and associative?

__ Yes. For instance the operation ______ on the set _______
__ No. Because ____________________________________

2. Is there any operation defined on a set S, that is neither commutative nor associative?

__ Yes. For instance the operation ______ on the set _______
__ No. Because ____________________________________

3. Is there any operation defined on a set S, that is commutative but not associative?

__ Yes. For instance the operation ______ on the set _______

__ No. Because ______________________________

4. Is there any operation defined on a set S, that is associative but not commutative?

__ Yes. For instance the operation ______ on the set _______

__ No. Because ______________________________

5. Label "true" or "false" and justify your choice of answer:
"A binary operation on a set S must be either both commutative and associative or neither commutative nor associative".

True / False. Because ______________________________

Intervention

Following the pre-assessment, the teacher, who was one of the researchers, guided a two stage class dialogue:

Stage 1 analyzed the commutativity and the associativity of the basic four operations on the rational numbers. The prospective teachers observed that addition and multiplication are both C&A, whereas subtraction and division are both NC&NA.

Stage 2 analyzed, analogously, the following four operations:

1. Concatenation: For any ordered pair of natural numbers (a,b), a*b=ab where the symbol ab designates a number which is formed by dovetailing "b" to the right of "a".
2. Seconding: For any ordered pair of integers (a,b), a*b is defined as b.
3. Arithmeaning: For any ordered pair of rational numbers (a,b), a*b=(a+b)/2.
4. Bisecting: For any ordered pair of points on the line (A,B), A*B is the midpoint of the line segment AB.

The prospective teachers were guided to observe that operations 1 and 2 are NC&A; operations 3 and 4 are C&NA.

Post-assessment instrument

Prospective teachers' ability to determine the logical independence of the two laws after this exposure was assessed through the following take-home assignment:

1. Give two examples of operations that are commutative and not associative, if any exists.

2. Give two examples of operations that are not commutative and associative, if any exists.

Results

Pre-assessment results

As expected, all 18 participants correctly claimed that C&A and NC&NA operations exist, and were able to give adequate examples of such operations (e.g., addition of natural numbers for C&A operations and subtraction of rational numbers for NC&NA operations).

Six prospective teachers (33%) correctly argued, prior to the intervention, that C&NA operations exist, but only three of them gave adequate examples of such operations (arithmeaning and a*b=-(a+b)). The twelve (67%) who argued that C&NA operations do not exist, stated that commutativity implies associativity. Five of them further explained that commutativity is a more general law than associativity since commutativity means changing order, and in the specific case of associativity the change is in the order of the operations.

Four prospective teachers (22%) knew, prior to the intervention, that NC&A operations exist. One of them gave a specific, adequate example of multiplication of matrices, two stated that "some groups are associative but not commutative" and one gave no example of NC&A operations. All but one of these four prospective teachers also claimed that C&NA operations exist. Of the 14 (78%) prospective teachers who argued that NC&A operations do not exist, ten stated that "associativity implies commutativity". A typical argument, used in several minor variations by the other four prospective teachers, was: "The associative law refers to three elements, whereas the commutative law refers only to two elements. Therefore, the commutative law is inferior to the associative law. Hence if the stronger, associative law holds, an inferior law must hold too."

Eleven of the 18 participants (61%) explicitly stated, in response to question 5, that all binary operations are either both commutative and associative or neither commutative nor associative. Six of them commented that this claim is based on their experience with operations. Three (17%) argued that binary operations can be C&NA but not NC&A and one (5%) argued that binary operations can be NC&A but not C&NA. Only three prospective teachers (17%) correctly claimed that the two laws are independent. These were the prospective teachers who argued, in respond to the previous two problems, that C&NA and NC&A operations exist.

Post-assessment results

After the intervention, all but one of the prospective teachers were able to generate two adequate examples of C&NA operations. The exceptional prospective teacher gave one adequate example and one inadequate example of C&NA operations. With respect to NC&A operations, nine prospective teachers provided two adequate examples of NC&A operations. Seven gave only one adequate example, one gave an adequate example and an inadequate example, and one gave no examples. The apparent difference between the prospective teachers ability to generate C&NA examples and NC&A examples was evident not only by the fact that more of them gave two adequate examples of C&NA operations, but also in the variety of the examples given. Altogether, nineteen different examples of C&NA operations and only five different examples of NC&A operations were given. In the conference we shall discuss these differences and relate them to those found in the pre-assessment.

It is noteworthy that in the post-assessment, all but one of the participants were able to provide at least one adequate example of both C&NA and NC&A operations. The wide variety of examples, to be presented in the conference, supports the claim that a major change occurred in these prospective teachers' perception of the logical relationship between the two laws.

Discussion and Implications

Our results indicate that the tendency to view the commutative and the associative laws as mutually dependent is very recurrent among prospective junior high school teachers specializing in mathematics education. It seems that these prospective teachers, who received a rather intensive instruction in mathematics, are still influenced by the initial, prototype examples of the basic mathematics operations, each of which is either C&A or NC&NA. Since today's prospective teachers are tomorrow's teachers, they may inadvertently perpetuate this misunderstanding to their future students. Thus, it is imperative to help them develop an accurate conception about the relationship between commutativity and associativity.

Our results also indicate that a relatively short and simple intervention, based upon a few examples of operations that are either

C&NA or NC&A, can be quite effective in raising prospective teachers' awareness of the independence of the commutative and the associative laws. As a result of the intervention, they were able to generate examples of C&NA and NC&A operations. The importance of such an activity was emphasized by Bartina (1986).

Beyond the issue of the independence of the commutative and associative laws, it seems that this activity raised the prospective teachers' awareness of the importance of counter examples in the teaching/learning process and encouraged them to analyze the sources of their own misconceptions. The following quotation, taken from a preservice teacher' assignment, is one example.

> "Although I know that commutativity and associativity are not dependent, I still find it hard to believe that this is so. My doubts are probably due to the fact that for a long time I have been confident that these two laws "go together". In order to convince myself of the contrary I am still trying, in every spare moment, to find examples of C&NA and NC&A operations."

References

Bell, A., Fischbein, E., & Greer, B. (1984). Choice of operation in verbal arithmetic problems: The effects of number size, problem structure and content. *Educational Studies in Mathematics* , *15* , 129-147.

Bratina, T. A. (1986). Can your students give examples? *Mathematics Teacher*, *79* , 524-526.

Greer, B. (1987). Nonconservation of multiplication and division involving decimals. *Journal for Research in Mathematics Education*, *18* , 37-45.

Hadar, N., & Hadass, R. (1981). "Between associativity and commutativity". *International Journal of Mathematics Education in Science and Technology*, *12* , 535-539.

Hart, K. (Ed.). (1982). Children's understanding of mathematics: 11-16 . London: John Murray.

Swan, M. (1983). *Teaching decimal place value: A comparative study of "conflict" and "positive only" approaches.* Nottingham, England: Shell Centre for Mathematical Education.

Tirosh, D., Hadass, R., & Movshovitz-hadar, N. (1990). Preservice teachers thinking about the distinction between two basic laws in mathematics. *Proceedings of the International Conference on Teachers thinking* (442- 447). Beer Sheva, Israel: Ben Gurion Univ.

Tirosh,D., & Graeber, A. (1989). Preservice elementary teachers explicit beliefs about multiplication and division. *Educational Studies in Mathematics*, *20* , 79-96.

FIRST STEPS IN GENERALIZATION PROCESSES IN ALGEBRA *

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The importance of investigating and straightening the steps involved in the generalization process prior to its formalization in elementary algebra is stressed. The feasibility of approaching these intermediate steps is shown via an experiment with children 12-13 years old.

INTRODUCTION.

'Generalizing is the lifeblood of mathematics, and algebra is the language in which generality is expressed. In order to learn the language of algebra it is necessary to have something you want to say. You must perceive some pattern or regularity, and then try to express it succinctly so that you can communicate your perception to someone else ...' (Mason, 1985).

Within the mathematics secondary school curriculum algebraic language is used as a medium for expressing generality, but little or no attention is paid to the steps involved in generalization, prior to its expression. We generally pass quickly to the expression of a generalization, without having evidence that children have identified and grasped a pattern or a regularity. Algebraic language is linked to the expression of a generalization, but to reach this stage prior steps are necessary. Several hypothesis and findings concerning children's difficulties in formalizing algebraically generalization have been made ([1], [5], [7], [8]); we consider, however, that further research is needed for a deeper investigation of children's performance during the steps before formalization; we consider that difficulties for both generalizing and learning the algebraic language could be rooted in those previous steps.

Mason [7] identified four steps involved in generalization: 'seeing' - mentally grasping a pattern or a regularity; 'saying' - articulating the insight in words; 'recording' - using symbols to formulate the generalization; 'testing' - verifying the validity of the formulation. We will adopt this framework to study children's generalization processes in pre-algebra.

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BACKGROUND.

As reported in [9] we were studying how children beginning the secondary school (12 - 13 years old), that is children without any formal knowledge of algebra and algebraic language, generalize and formulate generalization. We found that when faced with tasks similar to those proposed by [3], [5] and [7] (Fig.1 is a representative example):

1) Apparently, the great majority of children 'saw' a pattern in the graphical sequence; we say *apparently*, because the only evidence we had for this was that they were spontaneously continuing the drawing. 2) By counting the appropiate elements (dots, sides, squares) some of them completed the corresponding number sequence; they used a horizontal strategy: from the drawing to the number. 3) There were also some who considered the number sequence as being another separate task (similar to what Dufour-Janvier et al. [2] report) and found or invented a rule to complete it; they generally failed in applying it when a discontinuity in the sequence appeared (e.g. Fig.1: passing from 5 --> to 10 --> instead of the subsequent step 6 --->). 4) Few children tried to generalize for big numbers: some of them ignored the drawings, switched to the number sequence, disregarded patterns and, using a proportional strategy, multiplied an already known result by an appropiate factor (e.g. Fig.1: to find how many sides 100 squares will have they multiplied 31, the number corresponding to 10 squares, by 10); others ignored the number sequence, disregarded patterns, and focusing on the single basic shape generalized directly from it (e.g. Fig.1: a square has 4 sides, 100 squares will have 400 sides). That is, in their attempt to generalize, children: a) disregarded all patterns and b) in general did not use both representations; only one of them was directing their attempt to generalize. 5) Some children tried to express the generalization algebraically, but were not able to express a pattern (e.g. Fig.1: Y squares will have X or Y sides, i.e. an unknown number of squares will have an unknown number of sides).

From the answers obtained three aspects struck our attention:

1) in their attempt to generalize children did not link the two proposed representations (the graphical and the numerical one), even if they were probably considering that there was a relation between them; in fact, they used the drawings to complete the number sequence;

2) they could not express a generalization;

3) they disregarded all patterns when trying to generalize.

In relation to the first point, a careful analysis of the proposed tasks (Fig.1) shows that for solving them it is necessary for the child to:

... realize that there are two representations (in two different languages) of the same problem;
2) be able to switch from one to the other;
3) grasp the rule (see the pattern) behind the growth of the shape sequence (graphic representation) and of the number sequence (numeric representation);
4) realize that both rules are the same rule;
5) have a language for expressing the rule just found in a synthetic way; in this environment it implied using the algebraic language, that is giving a third representation of the problem. These points (from 1 to 5) correspond to points of difficulties; as we have already pointed out, children could not link different representations. And we completely agree with Lesh et al. [6] when they point out that 'these "translation (dis)abilities" are significant factors influencing both mathematical learning and problem-solving performance, and that strengthening or remediating these abilities facilitates the acquisition and use of elementary mathematical ideas'.

Referring to the second point, the lack of a 'final result', a formulation of the generalization, except for showing that children could not do it, did not give additional information about WHY they could not do it. Of course, they lack an appropiate language for expressing a generalization; they could hardly use the literal symbols and it seems that the graphical and the numerical languages they could use, tied them to particular examples and perhaps inclusively acted as obstacles for the generalization. But possibly there were also other kind of obstacles related to stages previous to 'recording' a generalization. For example, from the results we could hardly even be sure that children had really grasped a pattern; it concerned a mental process we had no evidence about.

EXPERIMENT and METHODOLOGY.

We designed an experiment to get more information about: 1) if and how children were identifying patterns; 2) how the different representations were interacting. This concerned, using Mason's terminology, 'seeing' and 'saying' a pattern (first steps involved in the generalization process), not yet 'recording' a generalization. We decided to run the experiment in a LOGO environment taking into account that: a) LOGO could be an appropiate language for expressing a pattern, an identically repeating pattern formed by one basic element (e.g. Fig.3), using REPEAT; b) this use of REPEAT could act as a magnifying glass allowing us a clearer observation of *how* and *if*

children were identifying patterns. We worked with 30 children (12-13 years old, first year of secondary school), a subgroup of already tested ones [9]. They already knew how to write simple procedures in LOGO, without inputs, and they had just been introduced to the primitive REPEAT as a tool for compacting a sequence of repeating commands. The experiment was designed to last 3 sessions (50 minutes each) and it took place once a week during the children's normal computer workshop. They worked in pairs and cooperation between them was encouraged. During each session they were involved in specifically designed activities and a worksheet was given to them. Direct observation and worksheets filled in by children during each session were our data.

ACTIVITIES AND RESULTS.

First activity. Children were given an extended LOGO procedure (Fig.2), asked to run it, sketch the drawing it produced and then compact the procedure using REPEAT. We observed if and how they were identifying a pattern and how the interaction between the two representations took place. All children, without exception, completely ignored the drawing they got and concentrated exclusively on the analysis of the procedure. For almost all of them the perception of patterns evolved in the same way: 1) they saw the isolated repeating elements (FD 20 appeared 12 times, etc.); 2) after signaling the non-commutativity of the instructions they identified FD 20 and RT 90 as a block; 3) in order to see the four big repeating blocks they needed explicit indications. All of them used the primitive REPEAT to express the pattern found. During the whole activity they were spontaneously 'testing' their hypothesis in direct mode and the original shape worked as a verifier. At no moment did the drawing helped them for compacting the procedure. The only link between the two representations was that one (the shape) worked as a verifier of the other (the procedure). In other words:
1) seeing a pattern was not straightforward for them; 2) REPEAT was a good tool that helped them become aware of a pattern and allowed them to express it while remaining in the same representational system; for us it acted as a magnifying glass which focused on their way of proceeding; 3) children were considering only one representation when solving the problem but since they were constantly testing their hypothesis, they spontaneously established a first natural link between given representations (the extended procedure and the drawing) and between their own representation (the compact procedure) and the drawing.

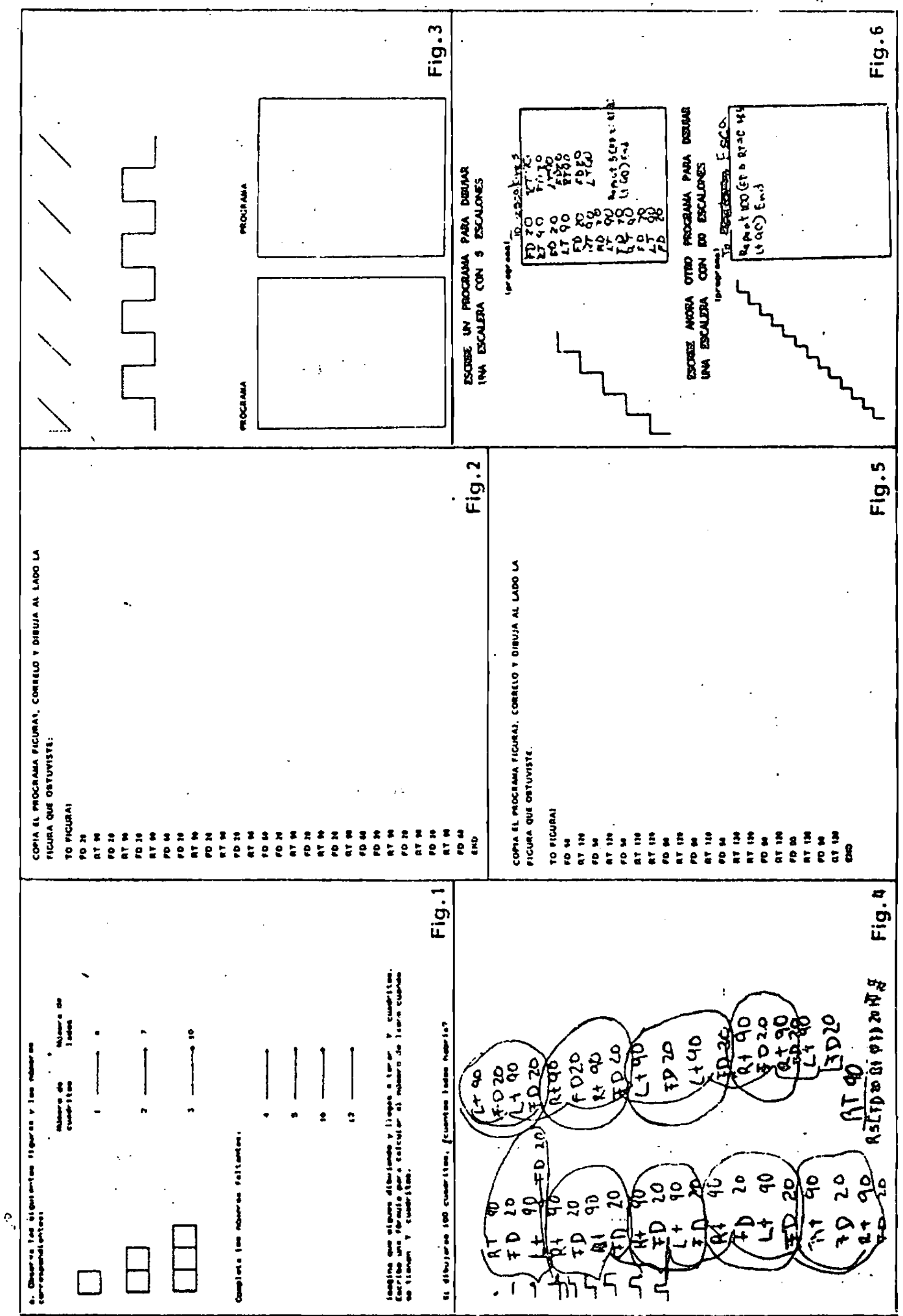
Second activity. Children were faced to geometric shapes (Fig. 3) and asked

to write compact procedures to produce them. None of them had any difficulty in identifying the pattern and continuing the drawing on paper, but none of them could write a compact procedure by observing the drawing. They could not 'say' the pattern. But the environment (LOGO) was offering them a step by step way of proceeding: they reproduced the drawing in direct mode and afterwards tried to identify the pattern comparing the figure they had obtained with the commands they gave. Figure 4 shows how a pair of children was looking for a pattern. These results show that: 1) when children needed to express a pattern in a representational system different from the one in which the pattern was given, they, first of all, tried to translate 'textually' what was given to the new representational system (from graphics to LOGO), preserving in that way all the characteristics and meaning of the problem; 2) perhaps because they were able to give, all by themselves, the new representation of the problem, the link between both representations was stronger; 3) once installed in the new representational system, the same in which they would express the pattern, they spontaneously used both representations to look for it; using Lesh et al. [6] terminology, we will say that they were performing 'transformations' within the new representational system, but at this stage 'transformations' and 'translations' were clearly interdependent.

Third activity. It was similar to the first one. Children were given a procedure (Fig. 5), asked to run it, sketch the drawing and compact the procedure. At the beginning of the task they were explicitly asked to try to use both representations for compacting the procedure. Once more, none of them considered the shape and analyzed only the given procedure. They had learned from the first activity and they could easily identify the big blocks. They were constantly testing and the shape played only the role of verifier.

FINAL COMMENTS.

A first conclusion is that having a medium for expressing the pattern they are seeing helps children become aware of it. If for 'saying' a pattern a different language is needed (as it was the case of algebraic language in the analized paper and pencil tasks), it seems that children need a scaffold: the possibility of translating the given situation to the new language while maintaining its original structure and meaning; when reaching this point, they start performing transformations within the new domain to approach the solution; it seems that when they construct the new representation by themselves, it is easier for them to spontaneously link it



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with the original one. It is only after gaining confidence about this way of proceeding that the process can be shortened and they can pass directly from the analysis of the shape to the expression of the pattern. Figure 6 shows this shortening. Later, after being introduced to the syntax for writing general procedures with one input, we could observe an analogous process: children used the same scaffolding strategy for writing general procedures. Given a shape they were asked to write a general procedure that maintained the given proportions. First of all they wrote a procedure using the given numbers, in this way passing from one representation to another; after that they generalized the procedure substituting the numbers that followed the primitive FD by the variable, operating it when needed and using the shape as verifier. Hoyles et al. [4] report a similar behavior emphasizing how a 'specific case served as a "concrete base" from which to generalise'. In our study, we could observe that after some such experiences children did not write particular procedures anymore and were able to directly write general ones.

Although algebraic language is the most appropiate language to express generality in mathematics, its use does not seem to encourage the development of the intermediate processes involved in generalization; in particular those processes involved in 'seeing' and 'saying' a pattern or a regularity. This characteristic becomes crucial at the beginning of the study of algebraic language, as was the case of the tested children. A tension between algebraic language, as a medium for expressing generalization, and the impossibility of using it, because it is not yet installed, arises. Hence the necessity of looking for other media that will encourage the development of the already mentioned steps and allow their expression. Not straightening these steps could be a source of difficulties for both the development of generalizing abilities and the installation of the algebraic language. This experience shows the feasibility of approaching these intermediate steps.

REFERENCES.

[1]BOOTH,L.R. (1984).'Algebra:Children's Strategies and Errors. A report of the strategies and errors in secondary mathematics project', NFER-NELSON.

[2]DUFOUR-JANVIER,B.,BEDNARZ,N. and BELANGER,M.(1987). 'Pedagogical Considerations Concerning the Problem of Representation', in Janvier,C.(Ed.), Problems of Representation in the Teaching and Learning of Mathematics, LEA.

[3]GEOFF GILES, DIME Pre-Algebra Project, 'Number Pattern 1-Simple Mapping', Department of Education, University of Stirling.

[4]HOYLES,C.,HEALY,L. and SUTHERLAND,R. (1990).'Patterns of Cooperation and Discussion between Pupil Pairs in Computer and Non-Computer Environment', (unpublished paper).

[5]LEE.L. and WHEELER,D. (1987).'Algebraic Thinking in High School Students: their Conceptions of Generalization and Justification', Department of Mathematics, Concordia University, Montreal.

[6]LESH,R.,POST,T. and BEHR,M. (1987).'Representation and Translations among Representations in Mathematics Learning and Problem Solving", in Janvier,C.(Ed.), Problems of Representation in the Teaching and Learning of Mathematics, LEA

[7]MASON.J. et al.(1985).'Routes to/Roots of Algebra', Center for Mathematics Education, The Open University, Great Britain.

[8]MATZ,M. (1982).'Towards a Process Model for High School Algebra Errors', in D.Sleeman and J.S.Brown (Eds.), Intelligent Tutoring Systems, London, Academic Press.

[9]URSINI,S. (1989).'Generalization Processes in Elementary Algebra: Interpretation and Symbolization',Proceedings of the XIV PME Conference, Vol. II, Mexico.

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Translation Processes Solving Applied Linear Problems

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Abstract

As part of a large research project 'Heuristic Education of Mathematics', developing and investigating strategies to teach applied mathematical problem solving, we inquired into the question of the transfer of the knowledge and skills from the domain of linear functions to real-world problems. Surprisingly students often used informal methods not taught in their lessons. After a full year of teaching mathematics, included a lot of applied problem solving, we established a shift from informal methods to the analytical (expert) solution method. There was also a significant difference between the learning results of three teaching strategies.

The educational setting

The mathematics curriculum in the Netherlands is changing the last 5 years, moving from stressing pure mathematical theory and techniques to using real world situations and applied problems as an important part of the national curriculum. At first the curriculum of the higher levels of the secondary school (grades 10-12, age 15-18) was reformed (De Lange 1987) with applied calculus, statistics and discrete mathematics as the main topics in the new subject 'Mathematics A' (grades 11-12). In grades 7-9 the Dutch mathematics curriculum is yet rather traditional, emphasizing algebra, factoring, solving equations, sophisticated definitions and notations of functions and so on. Making sense of functions in real-world situations, in science and economics and so on students have to achieve the goals as formulated in the NCTM-STANDARDS (1989, p. 154): In grades 9-12, the mathematics curriculum should include the continued study of functions so that all students can-

- model real-world phenomena with a variety of functions;
- represent and analyse relationships using tables, verbal rules, equations, and graphs;
- translate among tabular, symbolic, and graphical representations of functions;
- recognize that a variety of problem situations can be modelled by the same type of functions.

The main research problem to discuss in this paper, is the question of the transfer of the knowledge and skills from the domain of linear functions the students had acquired in grades 7-9. This means we 're interested in the extent to which the students will succeed in utilizing their knowledge and skills in other situations than those of the context they have been taught (Van Streun 1982). In grades 7-9 the context dealt only with f,x,y and straight mathematical situations whereas in grade 10 the applied functions and situations were very important in the applied calculus course. To describe some important problem solving behaviour in the domain of functions we use the diagram of Janvier (1987, p. 28) about translation processes. By a translation process, he meant the psychological processes involved in going from one mode of representation to another, for example from an equation to a graph. In the diagram the modes of representation of variables is limited to four, namely, verbal description, table, graph, and formula (equation).

Diagram 1. Translation processes.

From \ To	Situations, Verbal Description	Tables	Graphs	Formulae
Situations, Verbal Description		Measuring	Sketching	Modelling
Tables	Reading		Plotting	Fitting
Graphs	Interpretation	Reading off		Curve fitting
Formulae	Parameter Recognition	Computing	Sketching	

Translating a problem from one mode of representation to another is an important heuristic problem solving method, especially in applied mathematical problem solving. As Silver (1985, p.262) pointed out we need to examine how to teach students better representation skills, embedded in a domain like (linear) functions. Looking at the diagram of Janvier we concluded that in the Dutch grades 7-9 the analytical and graphical representations of a linear problem (function) are playing a major role, while the verbal description of a linear situation is absent in the domain of linear functions. In advance of our research we stated that these absence of verbal described situations

could be the major obstacle to bridge the gap between the two different oriented curricula namely the formal curriculum of the grades 7-9 and the more applied curriculum of grade 10-12.

An important aspect of the problem solving process is the recognition of the same fundamental mathematical component in different problem situations (Krutetskii 1976). Janvier (1978, 1981) notes that the recognition of mathematical concepts in realistic problem situations - such as play an important part in the teaching of our research project - requires an ability to abstract fundamental mathematical components that can only gradually be developed. He assumes that such realistic problem situations make a strong demand on the pupils' verbal abstract intelligence. Silver (1979) found that successful problem solvers were far more likely than unsuccessful ones to relate and categorize mathematics problems on the basis of their underlying similarities in mathematical structure. Unsuccessful problem solvers were more likely to rely on surface similarities in problem setting or context, or the question asked in the problem, in judging problem similarity.

The pretest

As part of a comparative teaching experiment 420 students in 21 classes from 7 schools made a pretest at the end of the 9th grade. All students followed in 7-9 grades the described traditional courses in algebra and functions. Linear problems were part of the pretest. We tested the ability of setting up a formula when a graph, a verbal presentation or the coordinates of four points are given. Besides that we investigated the use of translation skills in solving linear problems. Regarding our main research question we decided to investigate the extent to which the students succeeded in utilizing their translation skills in setting up a formula to solving applied linear problems. A number of tasks from the pretest had to do with first degree inequalities and we repeat two of them in the posttest. We have listed three pretest tasks here.

Pretest, task 15.

An electricity company offers a choice between two tariffs: $K(v)=46000+16v$ and $T(v)=20000+16v$ in cents per year; v represents the amount of kWh used up.

How many kWh you need to use up for tariff K to be cheaper than tariff T?

Pretest, task 16.

A car with a petrol engine costs 4000 guilders in yearly expenses (tax, insurance, etc.). Other expenses (such as petrol) come to 0.35 guilders per kilometre. A car

with a diesel engine costs 6000 guilders in yearly expenses and other costs come to 0.19 guilders per kilometre. How many kilometres do you need to drive in order to render the diesel engine cheaper?

Pretest, task 21.

A train leaves Paris from Lyon at a speed of 120 km/hour. An extremely fast train leaves a quarter of an hour later at a speed of 200 km/hour.

At what distance from Paris will the fast train overtake the first train?

We expected that in task 15 the students would quickly recognize the inequality and would use the analytical method for the solution. The jump from an inequality to task 16 is greater because the formulas are not supplied, so we expected less transfer. The solution of task 21 using an inequality demands a problem transformation which would probably be out of reach for many students, because the structure of the formulas is not clearly visible, as it was in task 16. Judging from the following table of percentages of (mostly) well executed tasks this expectations seems to tally.

Table 1. Percentages of good solutions of linear pretest tasks.
The total number of students is 420.

Task	1	15	16	21
Percentage good solutions	82%	46%	48%	25%

Surprisingly, however, task 16 turns out to have been no less well than task 15 whereas we had expected that the pupils would fair distinctly worse with task 16.

The question now is: which methods of solution did the pupils in fact use after three years of formal mathematics education in the secondary school? What was the transfer of training in analytical methods and algorithms? Table 2 states the comparative use of the solution methods on the correctly solved tasks, classified in terms of the four following solution methods. TAR means 'Translation into an analytical representation', TGR means 'Translation into a graphical representation', SNA means 'Systematical Numerical Approach' for example in the form of a table and MEA means 'Means-Ends Analysis', the process of reasoning in terms of the situation eliminating the difference between the given situation and the aim.

Table 2. Comparative use of methods in correct solutions of pretest tasks.

Task	Methods			
	TAR	TGR	SNA	MEA
1	99%	0%	1%	0%
15	87%	0%	7%	6%
16	73%	0%	8%	19%
21	19%	12%	50%	19%

Looking at the percentage of correct solutions using the expected analytical method (TAR) we get table 3.

Table 3. Percentage of correct solutions in pretest tasks, using TAR.

Task	1	15	16	21
% TAR	80.2	40.0	35.0	4.8

In the first place we note that the choice of a solution method heavily depends on the task. The assumptions concerning the 'distance' between the formal inequality and the applied assignments turn out to be correct. The greater the 'distance' is, the less often the pupils use the analytical method. The surprising success in assignment 16 compared with assignment 15 is due solely to the application of another solution method than the expected TAR.

Comparing these results with the success percentages of the tasks testing the translation skills of setting up a linear formula it is evident that the heuristic detour of making a graph (TGR) could help many students, but nearly no one used a graph. That's why we stressed the heuristic method of drawing a graph in the experimental teaching of the 10th grade.

Not the ability to set up formulas in task 16 appeared to be the major difficulty (see 88% score in task 8) but the insight to use formulas to solve the linear problem. Not the translation skill itself but the ability to plan the solving and to choose a suited method from the long term memory seemed to be the major obstacle in solving task 16. It is also remarkable that students invent their own 'common sense' or heuristic methods when confronted with new problems like task 16 and task 21. Regarding similar patterns in the observed classroom activities of the experimental classes of the 10th grade we developed a problem-guided course for grades 7-9 with a hierarchical ordered sequence of methods, starting with common sense reasoning like MEA, followed by table work (SNA), by drawing graphs (TGR) and at last by setting up formulas (TAR) and solving inequalities or equations.

The comparative teaching experiment

In the comparative teaching experiment we put our product **HME** (Heuristic Mathematics Education) to the test by comparing the learning results achieved using our teaching material with those achieved with the use of the two other teaching variants **FMTA** (First Mathematics Then Applications) and **LDRC** (Learning by Discovery from Realistic Contexts). The design and the general results of that experiment are reported in Van Streun (1989,1990). The three teaching variants did

not differ from one another as regards subjects studied, number of lessons (100) or nature and number of tasks and problems. There was, however, a distinct difference in the order of practical applications and 'straight' mathematical tasks. In LDRC there is continual alternation, in FMTA the 'straight' tasks come first and then the practical applications, whereas HME features phased alternation. FMTA pays no attention to heuristic methods like TGR, SNA and MEA, LDRC only indirectly in a variety of situations and HME does do specifically using them as a way to grasp TAR.

The linear problems of the posttest

Most of the posttest tasks deal with calculus, chances and periodic functions and exponential growth. Two posttest tasks (1 and 8) were linear problems analogous to two tasks (16 and 21) of the pretest.

Posttest, task 1.

In a certain industry they need a truck for at least a year. They have the following choice: to buy a truck or to rent a truck. If they buy a truck, they will have to pay 15000 guilders in overhead expenses (interest, maintenance, tax etc.) and they will have to pay 1.50 guilders for every kilometre they drive. If they rent a truck, they will have no overhead expenses (they will pay by the lessor) and they will have to pay 2 guilders for every kilometre they drive.

How many kilometres do they have to drive per year to make buying cheaper than renting?

Posttest, task 8.

A jogger leaves at 9 o'clock in the morning for a long run. He keeps up a constant speed of 120 metres per minute. Another jogger leaves from the same starting point at 9.15 and follows the same route. He maintains a constant speed of 200 metres per minute.

At what distance from the starting point does the second jogger overtake the first?

Table 4. Percentage good solutions of analogous problems.

Task	HME	LDRC	FMTA
Pretest 16	52%	50%	37%
Posttest 1	92%	86%	82%
Pretest 21	25%	28%	24%
Posttest 8	71%	59%	49%

As expected after another year of mathematics education and the first struggling with a lot of real applied problems, the success percentages increased. From task 16 (pretest) to task 1 (posttest) the difference for each teaching variant is about 40%, while

the increase from task 21 (pretest) to task 8 (posttest) is 46% (HME), 29% (LDRC) and 25% (FMTA). After one year of teaching according HME the success percentage of the more difficult task 8 (posttest) increases significantly more than in the teaching variants LDRC and FMTA. Looking at the use of methods (table 5) the following facts strike us.

Table 5. Comparative use of methods in correctly solved analogous tasks.

Task	TAR	TGR	SNA	MEA
pretest 16	73%	0%	8%	19%
posttest 1	85%	2%	9%	4%
pretest 21	19%	12%	50%	19%
posttest 8	54%	12%	21%	13%

Although the graphic and numerical methods in particular just as common sense reasoning, had, in at least the two experimental teaching variants HME and LDRC, received more attention than had been the case in the first three years of secondary school, the greater success after the 10th grade is due primarily to the use of the analytical translation method TAR followed by application of the algorithmic technique for the solution of a linear inequality of equation.

Table 6. Percentage of correct solutions in analogous tasks, using TAR.

Task	HME	LDRC	FMTA
pretest 16	39%	38%	22%
posttest 1	81%	65%	57%
pretest 21	4%	7%	5%
posttest 8	39%	34%	23%

Table 6 contains per variant and per task the percentage of correct solutions in which the analytical method was used. From this we can immediately conclude that the extra learning gain displayed by HME in these compared pairs of tasks is largely due to the more successful application of the analytical method.

Not only the absolute but also the comparative use of TAR in correctly solved problems increases greatly. Split up over the teaching variants the comparative use of TAR for the mentioned first pairs of tasks (task 16 pretest and task 1 posttest) is 73%-88% (HME), 75%-75% (LDRC) and 60%-70% (FMTA). In the second two analogous tasks (task 21 pretest and task 8 posttest) the use made of TAR, split up over the teaching variants, increases as follows 15%-55% (HME), 25%-58% (LDRC) and 19%-46% (FMTA). Our conclusion from the analysis of these items is that, with the increase in the number of (mostly) correctly executed assignments in each variant we notice a shift towards the application of the analytical method.

Discussion

In FMTA the cognitive framework is built round the mathematical concepts and techniques which at first require all attention. This teaching strategy does not appear to favour the flexibility of mathematical knowledge because pupils become set in the mathematical concepts, rules and techniques with they are presented. Hen the cognitive schema is noted in terms of pure mathematics the relationship with the practical applications is more difficult to establish. The findings of our micro analysis of linear problems conform the general results of all the problems from the domain of applied calculus and statistics (van Streun 1990).

The problem-guided courses of LDRC and HME with metaphorical contexts which are fruitful in building up a flexible cognitive schema, effect a better learning result in solving applied linear problems, according the general results of the teaching experiment. The explicit attention for heuristic methods in HME and the explicit link between heuristic methods and their abridgement in an algorithmic tecnnique combined with special attention to the translation from a verbal description into an analytical representation appears to influence the ability to apply successful the analytical techniques to solve problems.

References

Janvier, C.: 1978, *Interpretations of Complex Cartesian Graphs*, Thesis University of Nottingham, Nottingham.

Janvier, C.: 1981, 'Use of Situations in Mathematics Education', *Educational Studies in Mathematics* 12, 113-122.

Janvier, C.: 1987, 'Translation Processes in Mathematics Education', in C. Janvier (ed.) *Problems of Representation in the Teaching and Learning of Mathematics*, Erlbaum, Hillsdale, N.J., 27-32.

Krutetskii, V.A.: 1976, *The Psychology of Mathematical Abilities in Schoolchildren*, University of Chicago Press, Chicago.

Lange Jzn, J. de: 1987, *'Mathematics, Insight and meaning'*, OW&OC, Utrecht.

National Council of Teachers of Mathematics : 1989, *Curriculum and Evaluation Standards for School Mathematics*, Author, Reston VA.

Silver, A.E.: 1979, 'Student Perceptions of Relatedness among Mathematical Verbal Problems', *Journal for Research in Mathematics Education* 10, 195-210.

Silver, A.E.: 1985, 'Research on Teaching Mathematical Problem Solving: Some Underrepresented Themes and Needed Directions', in E.A. Silver (ed.) *Teaching and Learning Mathematical Problem Solving*, Erlbaum, Hillsdale NJ, 247-266.

Streun, A. van: 1982, 'Heuristic Mathematics Education' in A. Vermandel, *Proceedings of the Sixth International Conference for the Psychology of Mathematical Education*, Universitaire Instelling Antwerpen, Antwerpen, 239-244.

Streun, A. van: 1990a, *'Comparing Strategies of Teaching Applied Problem solving'*, Department of Mathematics, Groningen.

Streun, A. van: 1990b, 'The Teaching Experiment Heuristic Mathematics Education' in G. Booker P. Cobb, T.N. de Mendicuti, *Proceedings of the Fourteenth International Conference for the Psychology of Mathematical Education*, Mexico, 93-99.

GRAPHICAL ENVIRONMENT FOR THE CONSTRUCTION OF FUNCTION CONCEPTS

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In this study an instructional intervention is described in which highschool students experienced certain function concepts in the context of a graphical environment which has characteristics of a "generic organizational system" (Tall, 1985). A second group of students was taught in a graphical environment without computers. The teaching strategies were based on a constructivist point of view and a inductive method was used to guide students in concept construction. No significant difference was found between the groups.

A common teaching strategy in mathematics is the use of graphical representations, mostly on the blackboard, but also on worksheets, textbooks, homework assignments or written examinations. Since microcomputers are more and more accesible, we have a new powerfull tool to represent graphs. This study is based on the development of graphical environments with computers which enable students to discover and acquire function concepts in Algebra and Trigonometry at the highschool level.

The approach subyacent to the teaching and learning strategies discussed here is based on a constructivist point of view which describes human beings as builders of theory and structures (Balacheff, 1990b, Schoenfeld, 1987).

Recent research on visualization is concerned with the effects of a visual versus a symbolic approach and how students relate both (Eisenberg Dreyfus, 1989). There are studies which show the positive effects of visualizing in mathematical concept formation (Bishop, 1989) and give convincing arguments for emphasizing visual components in the introduction of concepts in schools. But there are dangers in doing this carelessly because visual presentations have their own ambiguities (Goldenberg, 1988). The main

objective of the research described is to study the feasability and efficiency of a graphical environment (graphics program for the computer, study guide and/or tutoring by a teacher) in the construction of function concepts.

The function concept is a central one in mathematics because of its potential to tie together seemingly unrelated subjects like geometry, algebra and trigonometry. It is also a very complex concept which has several subconcepts associated (Dreyfus, 1990). Inspite of efforts to teach functions by means of multiple representations, highschool students show limited concept images of functions (Vinner, Dreyfus, 1989). The computer plays an important role in mathematics education, since it is considered a valuable tool to aid in the teaching learning process in mathematics (Wenzelburger 1990a). Tedious and complex computations can be done on the computer. The students remain free to concentrate on essential aspects of concepts. Carefully designed graphing software used thoughtfully presents new opportunities to teach functions successfully. Such software makes use of multiple linked representations (Goldenberg, 1988). Computer environments seem to be an ideal tool to build a curriculum from a constructivist point of view which allows students to make transitions between algebraic and geometric representations (Tall, 1985,1987). Pea (1987) puts computers in the context of interactive cognitive technologies. Computers can provide functional environment that promote mathematical thinking. They fulfill the process functions of being a tool to integrate different mathematical representations. It seems possible that building visual concept images can take students further and deeper into mathematics. Vinner, Herskowitz (1983) analyze concept formation and define concept image as the mental picture a student has of a concept,

this does not necessarily coincide with the mathematical definitions of the concept or the definition a students knows. A concept image includes some essential properties of examples and non-examples of the concepts. Dreyfus (1990) discusses advanced mathematical thinking and points out that an incomplete concept-image can turn into on obstacle for learning.

Herscovics and Bergeron (1984) describe a constructivist perpective of mathematics education which focuses on the student and the central question: how to guide students in the construction of mathematical structures which build upon available knowledge.

The graphical environment we describe below assume a constructivist point of view both for mathematics and its teaching. Such a cognitive approach is intended to induce meaningful learning by means of a "generic organisational system" (Tall, 1985).

The study guide was designed according to an inductive method which consisted of experimentation, observation, formulation of some hypothesis, further experiments to test the "hypothesis" and formulation of conclusions.

There were two types of interventions, (Treatment A, Treatment B), both based on a graphical environment.

Treatment A consisted of directed experiences with CACTUSPLOT, a function plotter which was available to the students together with a carefully structured study guide but also time for free exploration was given. The instructors intervention was held to a minimum. The mathematical contents presented in the main study were trigonometric function of type y=asin(x) and y=acos(bx). Students were supposed to learn about the role of a and b in the function equation and their effect upon graphs of these functions, their amplitude and their period. The graphical environment consisted in the use of CACTUSPLOT to graph the functions as propo-

sed in the study guide.

The hypothesis was that experiences with CACTUSPLOT and the study guide could directly stimulated the construction of the con-cepts of the trigonometric functions under study and that studen exposed to Treatment A would do better than students under Treatment B. Treatment B consisted in work with the studyguide and tutoring by a teacher. The graphing activites suggested by the study guide were done by the traditional paper and pencil method with tabulation or on the blackboord. The teacher's intervention was restricted to anwer questions since the students were supposed to learn by the guided discovery approach suggested in the inductively organized material.

A first pilotstudy was done in July 1989 with 29 highschool students in Grade 12 (Wenzelburger, 1989). The working hypothesis was that students could form the concepts under study: The role of a,b and c in determing the graphs of quadratic functions. There are complete data for 23 students (pretest, posttest) and it was observed that the students reached to objetives partially, especially those related to the interpretation of graphs.

A second pilot study was done in December 1979 with 27 highschool students in Grade 10 (Wenzelburger, 1990b). Each experience lasted six one-hour periods. Students were working in the computer center with CACTUSPLOT and a study guide on the shape and the real roots of quadratics functions.

The results show that some learning occurred, but the objetives were only reached partially.

In June 1990 a group of 60 highschool students, enrolled in a 12 th grade calculus class was randomly divided into two subgroups. There are complete data (pretest, posttest) of 49 students (28 of the students in Treatment A, 21 of those in Treatment B) The

experience lasted 3 and a half weeks (11 one-hour sessions). Treatment-A group had their mathematics class with the researcher in the computer center where each student had his own IBM compatible microcomputer, the other 30 students (Treatment-B-group) had their class with their teacher in the usual classrooom.

One the first day of the treatment students in both groups were given a carefully structured study guide based on the inductive method described above in accordance to the constructivist point of view expressed earlier.

A typical didactical sequence in the study guide is shown in Figure 1. The intervention of the researcher in the learning process was kept to a minimum. Questions answered usually referred to the handling of the function plotter or the computer. The didactical progression in the study guide was a follows: Amplitude of functions of type $y=a\sin(x)$, amplitude of functions of type $y=a\cos(x)$, frequency of functions of type $y=a\sin(x)$, $y=a\cos(x)$, amplitude and frequency of functions of type $y=a\sin(bx)$, and $y=a\cos(bx)$.

The main difference between Treatment A and Treatment B consisted in the availability of the function plotter and the computer. The Paper-and-Pencil group did graph the functions proposed in the study guide on paper or on the blackbard by means of tabulation. The teacher's intervention was restricted to check the graphs drawn and to answer questions about the way the study guide was written. No answers to the questions the students had to answer, were given by the teacher, but there was positive feedback referring to correct answers and interaction between students who compared their work. This interaction between students occurred much less in the computer group since each student had its own computer.

Day 1, Activity 1.
1. Students use CACTUSPLOT to graph functions like y=sinx, y=2sinx, y=3sinx, y=½sinx, y=-2sinx.
2. Students answer questions about each pair of functions and formulate a conclusion about the role of a in y=asinx.
3. Students experiment with more functions of type y=asinx.
4. Students reformulate the conclusions in 2. taking into account the definition of amplitude of a function.
5. Students indicate the amplitude of given functions without graphing.
6. Students are asked to write equations of graphs given in a file on a workdisk.
7. Students are asked to draw graphs without CACTUSPLOT of more functions of type y=asinx.

Figuere 1

The posttest consisted of 30 multiple choice questions and was a parallel form of the pretest. The closed test form was chosen to assure reliability (Splithalf, $r=0.90$, Treatment A group, $r=0.92$ Treatment B group) and to facilitate evaluation of the test results.

The questions were designed to measure if students had reached the main objetives proposed for the study guide. Table 1 shows the average percentages of correct answers by group.

Table 1

Average percent correct by group

	Pretest	Posttest
Treatment A (n=28)	31.07 (σ=10.13)	75.39 (σ=20.43)
Treatment B (n=21)	32.91 (σ=11.32)	75.52 (σ=19.89)

These results show that on the average the previous knowledge and the final achievement of both groups were almost equal. Both groups did learn about the same.

Average gain scores for both groups from the pretest to the posttest were calculated. No significant difference could be found ($t=0.814$, d.f.=47) between the gainscores.

We conclude that the use of a function plotter for a microcomputer did not make any difference in this research in the context of the graphical environment which was set up by the study guide.

graphing activities with paper and pencil or on the black board with the classroom teacher produced the same gain scores than graphing with the microcomputer and this inspite the fact that the Treatment B group did only graph about half of the functions the Treatment A group did. This can be explained by the fact that the graphing on paper probably was done more carefully and consciously than the graphing on the computer which required less effort and less intellectual involvement with the task at hand. The interaction with the function graphs was less extensive but more intensive for Treatment-B-group while the computer group was exposed to more function graphs in a more superficial way. Also the presence of the classroom teacher and group interaction in Treatment-B-group way have compensated for the absence of the computer and the graphics program.

A limitation of the present study is of course the posttest used which was of the closed type, very traditional and probably did not detect any real differences in concept construction which might have occurred between the two groups. Visualization and formation of concept images could have been quite different but this did not show in the test. Further studies are necessary or to corroborate the findings reported here or to shed more light on the advantages of the computer graphics environment. The way this was setup by the activities proposed in the study guide the computer was mainly used as an electronic blackboard. If more dyamic interactive graphing activities would have been suggested, a significant difference in concept construction may have been found.

An analysis of the errors on the posttest and the answers written in the study guides indicate incomplete or even deficient concept images as well as difficulties with the visual approach to tea--

ching which apparently did not lead to a good enough visualiza--tion.

These types of studies are in accordance with the lines of research described by Balacheff (1990a) referring to psychological studies which explore learning environment called microworlds. It is important to know more about how and what students learn in them in order to improve them.

Bibliography:

Balacheff, N.: Future Perspectives for research in the psychology of Mathematics Education. ICMI Study Series: Mathematics and Cognition. Cambridge University Press, 1990a.

Balacheff, N.: Beyond a Psychological Approach of the Psychology of Mathema--tics Education. Proceedings of PME 14, 1990b.

Bishop, A.I.: Review of Research on Visualization in Mathematics Education, Focus on Learning Problems in Mathematics, Vol. 11, No. 1, 1989.

Dreyfus, T.: Advanced Mathematical Thinking. ICMI Study Series: Mathematics and Cognition, 1990.

Eisenberg, T., Dreyfus, T.: Spatial Visualization in Mathematics Curriculum. Focus: On Learning Problems in Mathematics. Vol. 11, No. 1, 1989.

Goldenberg, E.P.: Mathematics Metaphors and Human Factors. The Journal of Mathematical Behavior. Vol. 7, No. 2, 135-173, 1988.

Herscovics, N., Bergeron, J.: A Constructivist vs. A Formalist Approach in the Teaching of Mathematics, Proceedings of PME 8, 1984.

Pea, R.D.: Cognitive Technologies for Mathematics Education in Cognitive Science and Mathematics Education. Schoenfeld (Ed.), Lawrence Erlbaum Associates, London, 1987.

Schoenfeld, A.: Cognitive Science and Mathematics Education. An Overview. Lawrence Erlbaum Associates, Hillsdale, p. 1-32, 1987.

Tall, D.: Using Computer Graphics Programs as Generic Organisers for the Concepts Image of Differentiation. Proceedings of PME 9, Utrecht, 1985.

Tall, D.: Algebra in Computer Environment. Proceedings of PME 11, 1987.

Vinner, S., Hershkowitz, R.: On Concept Formation in Geometry. ZDM, Vol. 1, 1983.

Vinner, S., Dreyfus, T.: Images and Definitions for the Concept of Function. Journal for Research in Mathematics Education, 20, 356-366, 1989.

Wenzelburger, E.: Cápsulas Gráficas para Microcomputadoras en Matemáticas. Memoria de la 3. Reunión Centroamericana y del Caribe sobre Formación de Profesores e Investigación en Matemática Educativa, San José, Costa Rica, Julio 1989.

Wenzelburger, E.: La Influencia de las Computadoras en la Enseñanza de las Matemáticas. Ciencia y Educación, Vol. 4, No. 2, Guatemala, 1990a.

Wenzelburger, E.: Computer Graphics for the Adquisition of Function Concepts. Proceedings of PME 14, México, 1990b.

The Potential for Mathematical Activity in Tiling: Constructing Abstract Units

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Abstract

The purpose of this research was to construct explanations of grade three children's activity in attempting to construct a tiling of the plane. Steffe, et al. have argued for a unitizing operation in numerical settings. Use of this unitizing operation in a geometrical setting is posited. We found instances of abstract geometric units in the reasoning of children in a tiling activity and suggest that the construction of abstract units is a pervasive mathematical activity transcending numerical settings.

Steffe, Cobb, and von Glasersfeld (1988) describe a unitizing operation used by primary school children in adding and subtracting whole numbers. A thesis of this paper is that the unitizing operation is not restricted to numbers but can be observed in geometric settings as well. Wheatley and Cobb (1990) describe the conceptual activity of young children engaged in construeting meaning for geometric shapes in a Tangram task. In this paper the mathematical activity of grade three children is described as they use a given shape to make a tiling of the plane. Particular attention is given to the cognitive operations used in attempting this task. Based on the analysis of young children's tiling activity, evidence for this thesis will be presented.

The unitizing operation is an important mathematical activity. Much of mathematics involves the construction of abstract units, whether it be whole numbers (taking six as a unit), fractions (units of one-third), decimals (units of one tenth) or measuring (taking a centimeter as a unit of measure) mathematics is constructing abstract units. The parallel between constructing and using units of geometric shapes in tiling closely parallels the construction of ten as a composite unit (Cobb and Wheatley, 1988). In this study, we noted a relationship between a child's ability to construct gestalt units from nonrectangular shapes and their use of ten as a unit in adding and subtracting whole numbers. We conjecture that unitizing (Steffe, Cobb, and von Glasersfeld, 1988) is a quite general and significant mathematical operation which transcends number.

The tiling activity

In the initial class lesson, each child was provided with a piece of square dot paper and asked to "draw a pattern that repeats." Some students saw this activity as drawing a picture on plain paper, ignoring the dots on the paper. Others drew a picture which was influenced by the presence of the dots on the paper. Some drew a single symmetric design. A few drew a design which they repeated horizontally across the page leaving space between each design. Few students interpreted the activity as a tiling.

In presenting the task, it is important to negotiate the conventions of interpretation so that the child attempts the task intended In this case, a story was told about Claxton the elephant winning a contest and getting a set of identical tiles The child was asked to draw a plan for the tile master to use in laying the tiles in their kitchen. After several days in which the grade three class worked on tiling activities, six children were selected for one hour clinical interviews using tiling and addition and subtraction tasks

As we observed the children initially engaged in an activity initiated by being asked to make a repeating pattern with a given shape, we were struck by the unexpected difficulty of the task. Naturally many of the children's drawings could be explained by their intentions Some students saw the activity as one of drawing a picture. Others were influenced by the dots on the paper they were given but set out to create a design. A few made a pattern that repeated but was not a tiling since the shapes were not contiguous but aligned across the page. After a period of negotiation, all students were attempting to construct a tiling. Based on an analysis of the children's drawings and video recordings of their activity, the following constructs were formed which proved useful in explaining the children's actions.

1. Construct an image of the shape.
2. Build a production procedure for drawing the shape on the dot paper.
3 Create a covering
4 Construct a repeating pattern.
5. Coordinate patterning and covering.

Each of these constructs are discussed below. The analysis was informed by the theoretical formulations of Kosslyn (1983) and Wheatley (1990). It will be argued that underlying each of these constructs is the construction of abstract units.

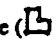
Construct an image of the shape

In order to proceed with drawing a copy of a shape, the child must first construct an image of the shape. It is possible to show a shape of sufficient complexity that no one could draw a replica of it without looking at the shape, part by part. The shapes presented in this study were relatively simple and most children could construct an image of the shape as evidenced by drawing it. One preschool child was shown a drawing of a right triangle and asked to draw it while it was in view. She shook her head and said, "I know it" but was unable to draw it. Her attempt resulted in an oval shaped closed curve. This child could distinguish triangles from other shapes and select a triangle to fit in a triangular region but had not developed the facility to draw it. There was evidence that she did "know the shape" even though she could not draw it. Drawing a picture requires eye-mind-hand coordination. Thus if a child is unable to draw a figure, it does not mean that an image has not been constructed. Figures can be so complex that an individual cannot construct a reliable image in a short period of time. For simple shapes, grade three children can construct reliable images but not for more complex ones. Some children drew a different shape than the one given because they had not constructed a vivid image of the shape (Poltrock and Brown, 1984).

In a task of continuing a tiling pattern composed of one-by-three rectangles which had been begun, Lauren constructed an L shaped figure formed by two of these rectangles and attempted to draw her construction. Questioning revealed that she did not see the L shape as composed of rectangles. She had not constructed the given shape (rectangle) but had formed a vague image of another configuration. Adrian began by drawing a given shape but was influenced by the space to be filled and began drawing shapes which were sometimes longer or of a different overall shape - an indication that the image of the initial

figure was not vivid.

Build a production procedure

In order to draw the shape in other positions on the dot paper the child had to construct a production plan for making the shape. For some of the children we interviewed, drawing each new shape was a problem solving task. Others soon developed a drawing plan and could quickly draw the shape in other positions and orientations. When students who did not have a production procedure attempted to create a tiling, their task was complicated by having to figure out how to draw the shape while for others this was easily done, allowing them to focus their attention on the relation of the shape to others. At times Jay had difficultly drawing the "Corner" or C Shape (). Unlike Lauren, he had not developed a production system for making the shape on dot paper; each time was a problem solving, trial and error process. He made many false starts. He appeared to be in the action (Schon, 1983) of drawing the C shape and was unable to reflect on his actions.

Covering

All students interviewed formed the intention of covering the page with the given shape. However, there were great differences in their activity. Lisa and Lauren easily drew the shape so as to achieve a covering. Adrian did not always achieve a covering (he left holes) although he seems to form the intention to cover. At times he lengthened or deformed the shapes to fit a space. In four of the five drawings Devonie and Candace placed the pieces to form a covering. In one case they each left a small square uncovered. In general, they were working for a covering. While having difficulty in drawing the shape and choosing a placement of the shape to achieve his goals, Jay was intentionally trying to achieve a covering and drew the shapes to fit nicely together if not always forming a regular pattern.

Constructing a repeating pattern

The task presented was to create a repeating pattern using just the given shape (a tiling). In constructing a repeating tiling pattern which can be extended indefinitely, the child must see a given shape as a part of a larger whole. The shapes taken together form a larger,

although unbounded, configuration which is a "unit." This unit is an abstract unit, i. e., a mental construction. The unit has been created when the pattern is conceptualized. Some persons may need to draw only a few shapes, others might need to draw many while certain individuals may be able to construct the pattern without drawing any shapes.

The students interviewed gave indications they were attempting to do this but at times lost sight of the goal. Lisa and Lauren each obtained a tiling using a rectangular pattern. They knew their pattern could be extended. None of the other four children interviewed obtained a tiling. Candace created several local patterns but they did not coordinate to form a tiling. Another student obtained a covering with the given shape but no pattern was formed.

Coordination of patterning and covering.

Three goals seems to guide the children's actions; making the shape, achieving a covering, and constructing a pattern. In order to create a tiling it is necessary to make a production procedure for drawing each shape and coordinate the separate acts of covering and patterning. Each of the acts necessary can be viewed as the construction of abstract units. Initially the child must take the shape as a unit and construct an image of it. Secondly, she must create a production plan. The activity of drawing the shape using a self-generated production plan can be considered as a unit of activity. That is, the action of drawing the segments constituting the figure is taken as a single action. Finally, in achieving a tiling the child constructs an abstract unit which defines the pattern. Any regular pattern can be thought of as a unit whether it has identifiable subunits or as one unbounded unity.

A frequent occurrence was the lack of coordination of the actions identified above. Several students became so engrossed in covering that any thought of patterning was lost. They were in the action of drawing a shape in a particular location to fill a space and could not distance themselves from their activity to consider how the shapes were relating to each other; for them it was a matter of fitting in the next shape. In some cases, children became

so engrossed in covering that they gave up the constraint of the given shape in order to fill a space and, in the process, drew other shapes. In other cases, children drew deformed shapes to extend their "pattern." Thus coordination of patterning and covering is a major task which many grade three children were unable to accomplish.

The construction of abstract mathematical units

When presented with the task of tiling with the C shape, Lisa immediately constructed a rectangular unit with two of these shapes and proceeded to use this abstract unit to make a tiling of rectangles which were then subdivided to form the C shapes given (see Figure 1).

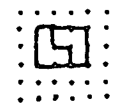


Figure 1. Lisa's rectangular unit

This action could also be taken as evidence of reversibility of unit construction; she could decompose the new unit into subunits. Thus Lisa performed a unitizing operation in constructing a rectangle by combining two of the given shapes. The rectangle was a new mathematical object of her creation. Lisa's intention in doing this was to form a gestalt which would be easy to work with, that is, shapes which would fit together easily in covering the plane. This action was possible only because she could distance herself from drawing the shapes and reflect on her activity.

Lauren constructed the given shape as a mathematical object and quickly developed a production pattern for drawing it. Her activity was deliberate and confident; there were no false starts or corrections. In the process of drawing the C shapes to form a covering which repeated, Lauren formed rectangular units from two C shapes as did Lisa but showed no evidence that she constructed the rectangles as mathematical objects. At no time did she draw the rectangle first and subdivide it as did Lisa nor did she given any indication of considering the rectangle as an object. The rectangles resulted from her production plan for making the C Shapes.

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Jay also formed a rectangle with the C shape but did not construct it as a mathematical object. In making his design, he formed three rectangles but they appeared in irregular arrangements and were not the only configuration formed: he also drew C shapes alone and unpatterned. Four other C shapes were drawn which were not in a rectangular pattern. On one occasion only, Devonie formed a rectangle with the C shapes but for her it was not an abstract unit. It is most unlikely she constructed an image of the rectangle formed by her placement of one of the shapes.

Candace clearly constructed a reliable image of the given shape; all 19 shapes she drew were congruent with the C shape given. She began with a diagonal and symmetric placement of the shapes which was potentially extendable. The third shape was placed in a position symmetrical to the second shape. However, the placement of her fourth shape (Figure 2) suggested that she did not construct the relationship of the pieces in such a way as to form an extendable pattern. In fact, for whatever reason, she failed to form a covering with the placement of the fourth shape. In all other placements she left no gaps. Following this drawing she began another local pattern extending downward. At this point she began "filling" this incipient pattern (Figure 3). In the process she created a rectangular form but there was no evidence to suggest she constructed the rectangle as a mathematical object; it resulted from her attempt to fill the spaces on each side of her vertical pattern and was not constructed as an abstract unit.

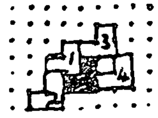

Figure 2. Candace A

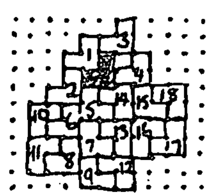

Figure 3 Candace B

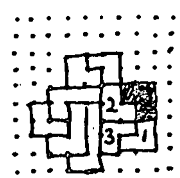

Figure 4. Adrian's Attempt

Adrian drew his first C shape below the given shape forming a new form with horizontal reflectional symmetry (Figure 4). He then attempted to "fill" the gaps created on the right. In so doing, the second shape he drew formed a rectangle with the given shape but Adrian did not construct it as a mathematical object. The next shape also formed a rectangle with his second shape but he did not intentionally place the figure to form the rectangle, the rectangle resulted from his action. In fact, he had difficulty drawing the third shape, having to correct his drawing. While Adrian formed some rectangles in the process of drawing shapes, they were not abstract units.

Summary

In this paper we have presented evidence of the pervasiveness of the unitizing operation in mathematics learning. In the tiling activity, students showed evidence of constructing several types of abstracts geometric units which paralleled the unit construction described by Steffe and his colleagues in a numerical setting. These constructions were associated with marked advances in their tiling activity.

Classroom activities which encourage the construction of units in a variety of settings are likely to be useful to students in coming to act mathematically. Tiling is a rich setting for developing the unitizing operation. The student is provided opportunities to construct an image of a shape, form a pattern with that shape and coordinate patterning and covering. Students are likely to benefit greatly in their mathematical development from opportunities to construct tilings of geometric shapes.

References

Cobb, P. and Wheatley, G. (1988). Children's initial understandings of ten. . Focus on Learning Problems in Mathematics, 10, 1-28.

Kosslyn, S. M. (1983). Ghosts in the mind's machine. New York: W. W. Norton & Co.

Poltrock, S. E. and Brown, P. (1984). Individual differences in visual imagery and spatial ability. Intelligence, 8, 93-138.

Schon, D. (1983). Educating the reflective practitioner: How professionals think in action. New York: Basic Books.

Steffe, L., Cobb. P., and von Glasersfeld, E. (1988). Construction of arithmetical meanings and strategies. New York: Springer-Verlag.

Wheatley, G. (1990). One point of view: Spatial sense and mathematics learning. Arithmetic Teacher, 37(6), 10-11.

Wheatley, G. and Cobb, P. (1990). Analysis of Young Children's Spatial Constructions. In L. Steffe and T. Wood (Eds.), Transforming children's mathematics education: An international perspective. Hillsdale: Lawrence Erlbaum Press.

THE EQUAL SIGN GOES BOTH WAYS. HOW MATHEMATICS INSTRUCTION LEADS TO THE DEVELOPMENT OF A COMMON MISCONCEPTION.

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Abstract

The premise of this paper is that we can help first grade students construct many relations and can help them to discover properties of relations by going through a rich variety of three levels of activities: (1) concrete manipulative-verbal level; (2) a level using diagrams and pictures, the perceptual-representation level and (3) an abstract symbolic level. At the end of the second grade tasks are presented to three groups of students: a group of students taught in first grade by a structuralistic mathematics program including the above mentioned activities; a second group taught by a mechanistic program; and a third group taught by a realistic program. The tasks presented measure student's understanding of the symmetrical property of the equivalence relation. The results show that the control group (the second and third group) ignores the bidirectional nature of the equal sign.

INTRODUCTION

In the mathematics program for students in first and second grade counting is done more than any other single activity. The thesis of this article is that although it is certainly important, counting is an insufficient vehicle for getting to know numerical and nonnumerical relations. The counting of the number of elements in a combined set leads to an operator notion of equality which emphasizes the result of an arithmetic operation (Kieran 1981). The operation involves the addition of two sets where the student combines the two sets and then counts the elements of the resulting set. It is the operator notion of equality, which is appealed to when the equal sign is introduced in school. Ginsburg (1983) points out, that primary school students interpret + and = in terms of actions to be performed. They read "3+5=8" as "3 and 5 make 8". A consequence of the student's interpretation of equalities in terms of actions is that he finds it difficult to read arithmetic sentences that do not reflect the order of his calculations, for example the sentence ?=3+5. The equal sign is viewed as an operator not as a relational symbol (Wolters, Timmermans & Smits 1989).

A common error is to ignore the bidirectional nature of the equals sign, thus producing expressions which, although joined by an equals sign, are not equivalent, but rather represent a procedural statement of how the problem is solved. e.g. "17+3=20+10=30+1=31" (Booth 1987). After primary school difficulties are still observed in students' understanding of equivalence (Kieran 1981; Filloy & Rojano 1985; Booth 1987, 1989). Developing a mathematical concept involves the development of a relational concept. In general, a relation is not a type of concept that we can show students directly. A student may place one block on top of another or see two sticks with one being longer than the other, but the concepts of "on top

of" and "longer than" are not in the blocks or the sticks. It is important to remember that generally in the development of a concept a sequence of steps is observed. We have to bear in mind that when we want to enable students to develop a notion about different relations several steps are required. First of all students need considerable time and opportunity to interact with materials. While they are doing this the students must be given the time and opportunity to verbalize the relations they observe. This manipulative-verbal level is a necessary step in the development of a relational concept. The second step is to verbalize the perceptual experiences of the relation. In this step the perceptual level is combined with the verbal activity. General terms like "bigger" can be adapted in the appropriate circumstances to mean longer, wider, thicker, taller, etc., when we are dealing with length. In other circumstances it might be developed into "holds more" or "covers more". The third and last step is that symbols are presented. This can only be done after the students have developed the corresponding underlying language structure. It makes a difference whether you ask a six year old child: "Which comes later (comes earlier) five or nine? or Which is bigger (smaller) five or nine?" (Fuson & Hall 1983). The meaning of a relation is made clear by doing and talking about what is done and seen.

The numerical relational concepts of "equal to", "more than" and "less than" are used to describe various qualities (length, density, temperature, or whiteness). Kagan (in Davydov 1978) distinguishes the following properties of the relational concepts: "equal to", "more than" and "less than".

1. At least one of the relations a=b, a>b, or a<b is true
2. If a=b is true, then a<b will not be true
3. If a=b is true, then a>b will not be true
4. If a=b and b=c, then a=c
5. If a>b and b>c, then a>c
6. If a<b and b<c, then a<c
7. Equality is a symmetric relation: if a=b then b=a.

In general two types of numerical relations are distinghuished: equivalence and order (nonequivalence) relations. The above mentioned properties of the three relational concepts show that order and equivalence relations are transitive. The property that does distinguish order relations from equivalence relations is that of being asymmetric: equivalence relations are symmetric (if 4+2=3+3, then 3+3=4+2) and order relations are asymmetric (if 7>4, then it is not true that 4>7).

We designed a teaching experiment to teach the concept of relation (in general, not just the equivalence relation) and the properties of a relation to a group of first grade students. Teaching 'Relations' is one part of an experimental structuralistic mathematics program. The other two parts of this program 'Operations' and 'Numeration Systems' are described in Wolters (1986a,b). The results of an evaluation of the complete structuralistic program on metacognition is described in Wolters (1988). In this article we describe first of all the teaching procedures to help students discover and construct relations and properties of relations. Secondly we describe the results of an investigation of the student's understanding of the equivalence relation at the end of the second grade. The latter investigation was also conducted with second grade students who were taught in a traditional way (mechanistic) or in a realistic manner (Freudenthal 1980).

METHOD

Subjects

169 second grade students comprise the sample for the study. The subjects were in second grade classes from five schools in different towns in The Netherlands. The schools represented a reasonable cross-section of city and rural school populations.

Procedure

The subjects followed one of three arithmetic courses: structuralistic course-condition (n=32); mechanistic course-condition (n=48); realistic course-condition (n=89). The subjects were tested in their classroom as a group. A paper and pencil task with three items were given to each student. For each item the experimenter repeated the printed instruction verbally as a classroom instruction.

Material

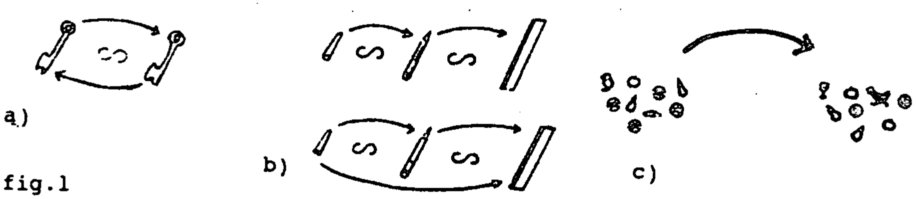
The structuralistic teaching course.

The teaching course is divided in three parts. The first part teaches relations and the properties of relations on a manipulative-verbal level. The second part deals with relations and properties of relations using models, diagrams and pictures. In the third part the same content is taught on an abstract, symbolic level. In each part the nonnumerical (physical and nonphysical) relations precede the numerical relations.

The manipulative-verbal stage
The teacher explains that objects go together to fulfill a purpose e.g. a cup and a saucer, a knife and a fork, a car and a garage, a bucket and a spade, a cricket bat and a ball, a doll and a doll's pram etc. The students have to verbalize for what purpose the objects go together e.g. a knife and a fork - to eat with etc. The teacher shows that there are many ways in which objects go together. For example, the teacher takes two objects of the same material (e.g. a teaspoon and a knife). With these two objects, a number of different verbal responses can be requested, each focusing on a different relation, e.g. they are made of the same material, they have the same colour, the knife is bigger than the spoon, and so on. Then a specific question leading to an examination of the symmetric property of the relation is asked: Is the relation still true if the objects change places? The answer is 'Yes' in case of the same colour and the same material and 'No' in case of the "bigger than" relation. For the transitive property we need three objects, e.g. a pencil, a teaspoon and a screw. The relation is "larger than". First the teacher examines this relation with the students for the pencil and the teaspoon; the answer is 'Yes', the pencil is larger than the teaspoon. For the teaspoon and the screw the answer is also 'Yes', the teaspoon is larger than the screw. The question to examine the transitive property is: Is the relation still true if we skip the middle object (the teaspoon)? In this case the answer is 'Yes', the pencil is larger than the screw. The same activities are gone through with regard to nonphysical relations such as "nicer than", "happier than" and "rhyme with".

In the same way the students discover that the relations "equal to" and "nonequal to" are symmetric and the relations "equal to", "more than" and "less than" are transitive. "Number" is only one of the qualities objects may have next to length, weight, volume and so on.

The perceptual-representation stage
As a general sign (symbol) to indicate that objects go together we use ∞. The symbol stands for: things go together or there is a relation between two objects. It indicates a relation but we do not know yet what kind of relation is meant: it is called a relation symbol. Students get the opportunity to verbalize and write down the specific relation between two given objects on worksheets. If a relation is symmetric (change places) it is marked by two arrows: one on top from left to right indicating that there is a relation and one on the bottom from right to left indicating that by changing the place of the objects the relation still holds (fig.1a). The transitivity property of a relation is marked by arrows as shown in fig.1b.



a) b) c)

fig.1

The result of a perceptual comparison of the number of objects in two different sets is registered by the symbols: =, ≠, > and <. So far the students did not have to use the symbols < and >, the symbols = and ≠ were sufficient. They are now required to define the relation by putting a symbol card in the proper place on a worksheet: the nonequal symbol is replaced by > or <. Finally they also have to indicate by arrows if the relation is symmetric and/or transitive (fig.1c).

The symbolic stage
First of all the idea that letters may stand for objects and object features is introduced. This is done by a memory game. Many different objects are placed on a table. Beside each object a lettercard is placed, which does not correspond to the first letter of the object. The students get one minute to memorize which letter goes with which object. After this minute the teacher removes the lettercards from the table, raises them one by one and asks to which object it belongs. The game is played several times with different objects and/or different lettercards and has two purposes: 1) to train the student's memory and 2) to learn to understand that letters in general can stand for objects or object features. The second step is: to use a letter in a relation instead of one of the objects. An example is:
rose ∞ e. The relation is "rhymes with". The question is, what can 'e' possibly be (nose, hose, etc.). To recapitulate what is learned in the previous two stages, the teacher asks the students to check if one is allowed to change places (symmetry and/or skip the middle object (transitivity). This step is varied by using different relations (nonnumerical) and putting

the letter at the left hand side. The third and last step is that the letters, the relation and the properties are given. The students have to tell what the letters may stand for, giving not only one but more possibilities. They learn to verbalize and symbolize their reasoning in the following way: example d ∞ a ∞ e, the relation is "rhymes with". As is indicated changing and skipping is allowed, so: d ∞ a, thus a ∞ d; a ∞ e, thus e ∞ a, and d ∞ e.
The relation between two or three numbers (<10) is notated by the symbols =, >, and <. Example: 2 ∞ 5 ∞ 7 gives 2 < 5 < 7, change is not allowed, skipping is allowed, thus: 2 < 7. It is easy to see after the nonnumerical experience, that in the same way letters may be used to stand for numbers. Example: f > s > u the question is, what numbers can be used in this formula. To make this stage a bit more challenging, we use problems like: 4 > e > 2. What number is "e"? To solve these kinds of problems we teach them to use the vertical numberline already known to them. In the example: 9 > z > 4 "z" stands for a range of numbers, which is easier to tell with this vertical number line. z > 4 means 5 and up; 9 > z means 8 and down, thus z = 5,6,7 or 8.

The task

The three items of the task are shown in fig.2a,b and c. The instruction for item 1 is: circle as many sums as you can find, horizontally as well as vertically. One sum is already circled. The instruction for item 2 is: put the proper symbols on the places of the dots. The instruction for item 3 is: make as many number sentences as you can with the numbers 37, 23, and 14. The numbers go into the rectangles and the =, + or - go into the circles.

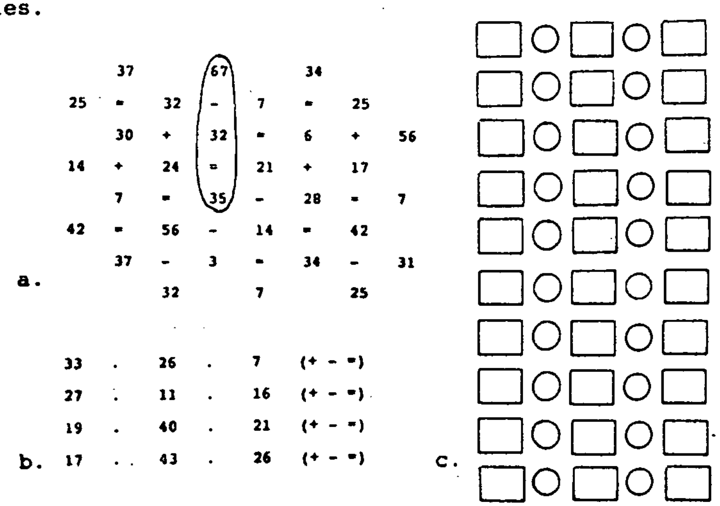


fig.2

Item 1 requires a passive use of knowledge; recognition of a correct addition or subtraction sentence is asked for. Item 3 requires an active use of knowledge; production of a correct addition or subtraction sentence is asked for. Item 2 is somewhere in between item 1 and 3.

RESULTS AND DISCUSSION

Number of correct answers

The total number of correct answers on all three tasks is presented in table 1. A T-test analysis shows (table 1) that the differences in number of correct answers on the three items together between the structuralistic and mechanistic condition as well as between the structuralistic and realistic condition is significant. This is not the case for the difference between the realistic and mechanistic condition.

Table 1. Means, SD and T-value for the number of correct answers on all three items.

cond	mech (n=48)	real (n=89)	struct (n=32)	mech
mean	10.85	10.43	13.72	
SD	3.34	3.47	4.12	
T-value		.70	4.03	-3.28
p-value		.483	.000**	.002**

When we look at the number of correct answers for each item separately, a different picture emerges. The results of a T-test analysis for the number of correct answers on item 1 shows that none of the differences between the three conditions is significant. The results of a T-test analysis for the number of correct answers on item 2 shows that the difference between the structuralistic and mechanistic condition is significant, but the difference between the structuralistic and realistic condition is not significant.
The number of correct answers on item 3 are shown in table 2. A T-test analysis shows (table 2) that the difference between the structuralistic and mechanistic condition as well as between the structuralistic and realistic condition is significant. This is not the case for the difference between the realistic and mechanistic condition.

Table 2. Means, SD and T-value for the number of correct answers on item 3.

cond	mech (n=48)	real (n=89)	struct (n=32)	mech
mean	2.58	2.31	4.78	
SD	1.57	1.65	1.75	
T-value		.94	6.93	-5.72
p-value		.350	.000**	.000**

It seems as if item 3 is responsible for the difference between the conditions on the number of correct answers on all

items (table 1). Table 1 and table 2 have the same pattern. We have to take a closer look at the difference between item 3 and items 1 and 2. The activity required for carrying out item 1 is recognition; a passive use of knowledge. The activity required for carrying out item 3 is production; an active use of knowledge. To produce more than two or three number sentences with three given numbers (item 3), it is necessary to engage in actively manipulating properties of operations (+,-) as well as the properties of the equivalence relation. Let us have a closer look at item no.2, the item that psychologically stands between item no 1 and 3. Besides looking at the number of correct answers, we examened the response patterns that occur. The pattern with all answers correct is not frequently observed. A pattern that is observed frequently in the mechanistic condition is: a. 33-26=7; b. 27-11=16; 19-40=21; d. 17-43=26 (correct, correct, wrong, wrong). A pattern that is observed frequently in the other two conditions is: correct, correct, no answer, no answer. It seems as if these students prefer no answer to an incorrect answer.

The relation between item 1 and 3.

As we have seen before, item 1 requires a passive use of knowledge of the equal sign and item 3 requires an active use of the same kind of knowledge. It is therefore important to analyse the relation between item 1 and item 3. We analysed the relation by way of an analysis of covariance on the number of correct answers with '=' on the right hand side - that is in the regular position - with item 1 as a covariate. The question to be answered is: what is the relation between the number of correct answers on item 3 and the number of correct answers on item 1 with the equal sign in the regular position. The results of the analysis of covariance are shown in fig.3.

Results of an analysis of covariance

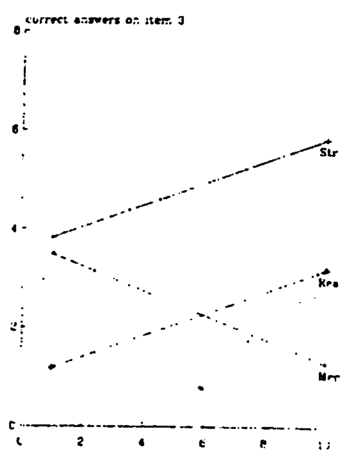


fig.3

For the mechaistic condition there is a negative (significant) correlation between item 1 and 3. The more correct answers there are on item 1 the fewer correct answers on item 3. For the realistic and structuralistic condition the correlation is positive and significant. But there is also a significant difference between these two conditions: the structuralistic condition gets higher scores on both items than the realistic condition.

The findings indicate that learning and mastery of the concept of equivalence are related to the type of instruction received. A mechanistic type of instruction does not help the students to integrate passive and active knowledge.
A realistic type of instruction, which is very popular in The Netherlands, offers more opportunity to integrate the two kinds of

knowledge.
But only the explicit emphasis on properties of relations (and operations) as is the case with the structuralistic type of instruction, gives ample scope for the students to show their abilities.

References

Booth L.R. (1987). Equations revisited. In: Proceedings of the 11th Annual Conference of the International Group for the Psychology of Mathematics Education. Montreal, p.282-288.

Booth L.R. (1989). Grade 8 students understanding of structural properties in mathematics. In: Proceedings of the 13th Annual Conference of the International Group for the Psychology of Mathematics Education. Paris, p.141-148.

Davydov V.V. (1978). The psychological characteristics of the prenumerical period of mathematics instruction. In: Kilpatrick J. & Wirzup I., Soviet studies in the psychology of learning and reading mathematics.

Filloy E. & T. Rojano (1985). Obstructions to the acquisition of elemental algebraic concepts and teaching strategies. In: Proceedings of the 9th Annual Conference of the International Group for the Psychology of Mathematics Education. Noordwijkerhout, The Netherlands, p.154-160.

Freudenthal H. (1980). Weeding and sowing. Preface to a Science of Mathematical Education. Dordrecht, Holland: Reidel.

Fuson K.C. & Hall J.W. (1983). The acquisition of early number word meanings. In: Ginsburg H.P. (ed.), The development of mathematical thinking. A.P. inc. London.

Ginsburg H.P. (1983). The development of Mathematical Thinking. A.P. inc. London.

Kieran C. (1981). Concepts associated with the equality symbol. Educational Studies in Mathematics, 12, 317-326.

Wolters M.A. (1986a). Is 89 larger than 91? In: Proceedings of PME 10. England, London 20-25 july, p.69-75.

Wolters M.A. (1986b). Rules in arithmetic. Learning the basic facts. In: Lowenthal F. & F. Vandamme (eds.). Pragmatics and education, p.261-273. Plenum Press, London.

Wolters M.A. (1988). Metacognition and elementary school mathematics. In: Proceedings of PME XII. Hungary 20-25 july.

Wolters M., Timmermans M. & J. Smits (1989). We know that already. Ideas of children about the '=' sign. Willem Bartjens 1, 35-41 (in dutch).

LEARNING IN AN INQUIRY MATHEMATICS CLASSROOM

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The investigation of children's mathematical learning in the classroom is fraught with difficulties. Neither a psychological nor a sociological perspective provides an adequate description of the complexity of the process. In addition, natural classroom settings which provide an opportunity to consider learning from both fields of view are generally not available. This paper is an investigation of the process of learning mathematics in a third-grade classroom in which problem solving, reasoning and communicating are emphasized.

It has long been recognized that children come to school with rich and diverse informal methods for solving mathematics problems. These methods have developed from children's experiences and reflect their current level of understanding. The role schooling plays has been to provide instruction on the more formal aspects of mathematics. However, many mathematics educators have contended that children's learning of formal methods would be greatly enhanced if mathematics teaching would recognize and build on children's constructions rather than imposing formal methods on them (NCTM, 1989). This focus would allow children to extend their mathematical learning in a relational manner (Skemp, 1976). While this position is generally acceptable, only a few classrooms exist in which teachers' actually do focus their instruction on students' developing mathematical conceptualizations (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Cobb, Wood, & Yackel, in press; Oliver, Murray & Human, 1990). In these classes, students are involved in activities that encourage them to reason mathematically and to communicate their thinking to others, rather than reproduce procedures provided by the teacher or textbook.

In addition, the majority of analyses that have been conducted on children's mathematical learning in the classroom have focused on the products of their learning as measured by achievement tests rather than on the processes by which students learn. The purpose of this paper is to present an analysis of the process of mathematical learning that occurs in a classroom in which learning mathematics through problem solving, reasoning, and communicating are of central importance. This investigation will consider mathematical learning from two distinct fields of view, that of psychology and sociology in an attempt to illustrate the contribution each conveys to an understanding of the process.

BACKGROUND AND THEORETICAL ORIENTATION

Traditionally, elementary school mathematics is a form of practice in which the emphasis is on memorization and reproduction of procedures presented by the teacher, and as such precludes situations which would encourage students' to communicate their reasoning. Consequently, opportunities to analyze children's learning in the ongoing natural setting of their classroom is lacking. Prior analyses of children's mathematical learning in which cognitive processes have been of central interest have either occurred in early childhood settings with primary caretakers (Saxe, Guberman, & Gearhart, 1987) or in experimental situations (Carpenter & Moser, 1983). Even the very carefully detailed cognitive models of children's arithmetical learning completed by Steffe (Steffe, Cobb, & von Glasersfeld (1988); Steffe, von Glasersfeld, Richards, & Cobb, 1983) were developed from the methodology of the teaching experiment in a one-on-one setting.

Reciprocally, several researchers have attempted to consider the possibilities for learning by considering the nature of the social interactions that occur in the classroom. These studies have considered the teacher/student interaction (Voigt, 1985) or cooperative peer interaction (Perret-Clermont, 1980). While these studies provide information about the nature of socially situated actions that constrain or enhance children's constructions, information on the cognitive processes is missing.

It seems what is needed is an approach to investigating learning in the classroom which would take into consideration both the cognitive and social processes involved (Cobb, 1990), and a classroom setting in which these processes could be analyzed. The theory of Piaget (1970) focuses on internal mental processes by which individuals construct knowledge as a result of their thought and action, while the work of Steffe and colleagues has extended this work to consider children's learning of arithmetical concepts in particular. The teaching experiment methodology used by Steffe in his research has enabled him to focus on the ways in which children give meaning to their experiences. From these conceptual analyses, he has developed detailed cognitive models to describe children's mathematical development.

The theoretical constructs derived from this research were used in this study for the cognitive analysis. The theoretical orientation of the symbolic interactionists Blumer (1969) following Mead (1934) emphasize the meanings that are negotiated between participants. Meaning is a property that arises out of the interaction that takes place among people in the course of their daily life. The meanings that are constituted are taken-as-shared and serve to constrain and enhance the meanings constructed by the individual. The research of Bauersfeld (1988) and Voigt (1985) extends this theory to classrooms to consider the nature of the patterns of interaction and the meaning that is interactively constituted among the participants which create opportunities for learning mathematics. This research has provided theoretical constructs that were used in the

analysis of social interaction in this study.

A central concept for both theoretical positions is the meaning that is constituted. Bruner (1990) has contended that the central concept of psychology is meaning and the processes and transaction involved in the construction of individual meanings. Meaning for the interactionist is also the central concept of sociology and the processes involved in the constitution of the fit of meanings attributed to the situation. This taken-as-shared meaning that is constructed through the process of negotiation of meaning forms the source for the analysis of social interaction, while individual meaning that is constructed through the process of reflective abstraction underlies the analysis of cognitive operations. The analyses of the meanings constituted in the intertwined patterns of cognitive action and social interaction that occur in classes in which reasoning and inquiry are emphasized provides an opportunity to investigate the complexity of learning.

Our classroom teaching experiments in second and third-grade has as a central goal the constitution of mathematical meaning. In these classes, students are involved in problem solving activities in which reasoning and communicating their thinking to others occurs in an atmosphere where inquiry is valued more than correct answers. Students are expected to explain and justify their thinking and to understand and question the reasoning of others (Cobb, Wood, & Yackel, in press). Consequently, the lessons and discourse that occurs serves as a source from which to conduct analyses. The method of analytical descriptive narrative described by Erickson (1982) was used to provide a method for presenting a coherent description of a sequence of lessons, while microanalyses of discourse (Voigt, 1985) was used to detail selected episodes.

ANALYSIS OF LEARNING IN A MATHEMATICS CLASSROOM

We are in the first year of a classroom teaching experiment in third-grade in which we collaborate with a teacher to construct instructional activities based on anticipated constructions children might make. In this class, the obligations and expectations that underlie the patterns of interaction have been mutually established by the teacher and students. Thus, the children are aware that during class discussions they are expected to explain and, if challenged, justify their solutions. They are also expected to try to understand others' explanations and ask questions for clarification. The teacher's role is to question, probe, and make suggestions based on the student's comments, rather than impose predetermined methods. The particular episodes that were selected occur over 4 lessons. The first 2 occurred on consecutive days, followed by the third two days later, and the fourth one month later. For the purposes of this paper the examples will not be presented in detail. The incidents that are described center around an activity, in which a 2-digit subtraction sentence is written on the board, which students are asked to solve without using paper/pencil. The intention of the activity is to encourage the construction of ten as an iterable unit

and nonstandard algorithms for subtraction (Cobb & Wheatley, 1988; Steffe, Cobb, & von Glasersfeld, 1988). Understanding ten at this level indicates a sophisticated level of understanding place value and has a powerful generalizing effect to understanding multiplication.

EPISODE 1

On this day the students were given the problem 61 - 37 = ___. The pattern of interaction that had been established was for students to give their answers and then to provide their explanations. The students had volunteered the answers of 24 and 36. Mark had given the answer 36 for which he explained, *"Well— 7 and 1, take away 1 from 7 equals 6. And um 3 take awa... 3 out of 6 equals 3."* "Then I got 36." Mark's explanation indicated that he viewed the problem as consisting of two parts to be acted on, starting with the ones. His solution suggests that he had not constructed units of one and ten as mathematical objects but instead was acting to manipulate written symbols. Following his explanation, Tzu gave her explanation which involved going through the tens and for which she arrived at 24. At this point, Jeff interrupted to say, *"Oh, I know how they did it! They switched it around. What Mark was doing was switching it around and took 1 from 7 and 3 from 6. All he was doing was switching it around."*

Mark made no comment, but looked intently at the problem and began to count on his fingers. The teacher continued and called on Rick who agreed with 24 and gave his explanation as taking 30 from 61 and subtracting 7 from that sum. As Rick finished his explanation, Mark interrupted excitedly:

Mark: Yeah! Now I understand!
Teacher: (still responding to Rick) Okay, you took 30 from 61.
Mark: (interrupting again) Now I understand!
Teacher: (intent on listening to Rick) Oh look what he is thinking that 61 minus 30 (writes 61-30=31)
Mark: (waving his hand interrupts again) Mrs. Frank! I know what I did!

Mark's initial explanation indicated that he thought of the problem as containing two parts, so many tens, so many ones. His procedure was to subtract the ones and then subtract the tens. In addition, he thought that subtraction, like addition, was commutative. Therefore, he rotated the digits in the ones column to resolve the dilemma of subtracting the larger number from the smaller number. This procedure was consistent with the way he had solved similar problems given in the clinical interview the first week of school. On addition tasks involving tens, he solved them using a combination of procedures of either adding tens, adding ones or the standard algorithm. His explanations of his methods suggested that he thought of tens as abstract composite units and counted representations (e.g., ten more).

Although Jeff challenged Mark's explanation, he was unable to offer any further justification. He counted to find the correct answer, realized his procedure was flawed, but did not have an alternative explanation other than counting. As he listened to Rick's explanation, he

seemingly reflected on his activity to reorganize his thinking and excitedly tried to tell the class. The teacher finishing with Rick's explanation asked, *"Is that the way you did it Mark"* Is that how you thought about it?" to which he nodded in agreement.

In this setting, learning opportunities were created as students were expected to negotiate their mathematical meanings. These meanings became the taken-as-shared knowledge that in, a context of recurrence, define specific mathematical practices.

EPISODE 2

On the next day, the class was again solving a problem involving decomposition. The problem was 74 - 37 = ____. Mark solved this problem using the procedure that Rick had used the day before of taking 30 from 74 and 7 from 34. On the next problem, 56 -28 = ___, he thought the answer was 32. Jeff again pointed out that he had done *"the same thing as yesterday."* Mark responded, *"Oh no! I don't think...I know. No I don't think its 32."*

Mark's ability to use Rick's method on the next day to solve a different problem suggests that he had reflected on his activity and had made an accommodation that he was able to use in a new setting. The fact that he solved the second problem using his old method indicates that this change was still temporary and subject to the situation. However, because the children were involved in discourse in which their ways of solving problems was of central interest, opportunities existed for them to become aware of the inconsistencies in their thinking.

EPISODE 3

The problem this day was 42 - 26 = _____. Alice had given the answer of 22, and was giving her explanation,

Alice: I took the 6 and the 2. And then I took away 2 from 6.
Mark: She's taking...turning around the numbers.
Teacher: Oh, Mark says you're turning around the numbers.
Mark: Ya. She is.
Teacher: So you took...you took the 2 from the 6?
Mark: (simultaneously) That's what I did the other day.
Jess: Oh! Oh! (frantically waving his hand) Mark's right. You can't do that Alice.
Alice: Why?
Jess: You can only do that in plus. You have to add.

Jess's comment reveals that he realizes that in subtraction "turning the numbers around" does not work like in addition. His insight provides a rationale for Mark's comments and further contributes to the meaning for subtraction being constructed by the members of the class.

Mark: She's subtracting 6 from 42. She's doing 2 instead of 2 take away 6. She's doing 6 take away 2 instead of 2 take away 6.
Teacher: Oh. Matt says you're taking this number(points to 6) from this number(points to 2), instead of this number(points to 6) Um...
Alice: But you can't...
Jim: But if you take the 20 from the 42, you've got 22

right there. And you just take away the 6 ones.
Teacher: Okay. You take 42 take away this 20?
Jim: You'd have 22 right there.
Teacher: You'd have 22 right there he says and you still have to take away 6. So he says that's why he didn't think it was 22.

At this point, Alice reconsiders and agrees that 22 is not a possible answer to the problem.

EPISODE 4

The problem on this day was 74 - 36 = ____. Some of the students have been thinking about solving this problem using negative numbers. The procedure they have been discussing is one which Scott gave which he first subtracted 30 from 70 and got 40. Then he subtracted 6 from 4 and got 2, and then wrote 40 - 2 = 38 to which Joel said, *"You got to get a negative 2, you put a minus sign next to the 2, so they know it."*

Having finished discussing Scott's solution, Mark then raised his hand to offer his solution.

Mark: I did it a different way. Cause you were going to ask that (meaning that he knew the procedure was for the teacher to ask for other solutions).
Teacher : Mark, you want to come up and show us?
Mark: (goes to the board). Well I did it. I had 74 take away 30.(He writes 74 - 30) and that equals 44 (writes 44). Then I had 44 take away 6 (writes 44) and that equaled 30, 38.(writes 38).

This episode occurred about a month later during which time this activity had not been used in the class. It would appear that solving problems in this manner had become routine for Mark and that he used it easily and was able to offer a concise explanation for his procedure. His method was accepted without question by the other members of the class indicating that for them the solution had also become agreed upon.

SUMMARY

The process of reasoning mathematically develops through participating in situations in which explaining and interpreting are part of the tradition. Classrooms which provide opportunities for students to communicate their thinking to others and to critique the thinking of others create situations in which they can construct mathematical meaning for themselves and participate in the mathematics of the community.

REFERENCES

Bauersfeld, H. (1988). Interaction, construction, and knowledge: Alternative perspectives for

mathematics education. In T. Cooney & D. Grouws (Eds.), Effective mathematics teaching (pp. 27-46). Reston, VA: National Council of Teachers of Mathematics and Lawrence Erlbaum Associates.

Blumer, H. (1969). Symbolic interactionism: Perspectives and method. Englewood Cliffs, NJ: Prentice Hall.

Bruner, J. (1990). Acts of Meaning. Cambridge, MA: Harvard University Press.

Carpenter, T.P., Fennema, E., Peterson, P., Chiang, C.P., & Loef, M. (1989). Using knowledge of children's mathematics thinking in classroom teaching: An experimental study. American Education Research Journal, 26, 499-532.

Carpenter, T.P., & Moser, J.M. (1983). The acquisition of addition and subtraction concepts. In R. Lesh & M. Landou (Eds.). The acquisition of mathematical concepts and processes (pp. 7-44). New York: Academic Press.

Cobb, P. (1990). Multiple perspectives. In L.P. Steffe, & T. Wood (Eds.), Transforming children mathematics education. International perspectives. (pp. 200-215). Hillsdale, NJ: Lawrence Erlbaum.

Cobb, P., & Wheatley, G. (1988). Children's initial understandings of ten. Focus on Learning Problems in Mathematics, 10 (3), 1-28.

Cobb, P., Wood, T., & Yackel, E. (in press). A constructivist approach to second grade mathematics. In E. von Glasersfeld (Ed.), Constructivism in mathematics education. Dordrecht: Reidel.

Erickson, F. (1982). Taught cognitive learning in its immediate environments: A neglected topic in the anthropology of education, Anthropology and Education Quarterly, 13, (2) 149-180.

Mead, G.H. (1934). Mind, self and society. Chicago: University of Chicago Press.

National Council of Teachers of Mathematics. (1989b). Curriculum and evaluation standards for school mathematics. Reston, VA: National Council of Teachers of Mathematics.

Oliver, A., Murray, A., & Human, P. (1990). Building on young children's informal arithmetical knowledge. In G. Booker, P. Cobb & T. de Mendicuti (Eds.), Proceedings of the Fourteenth International Conference on the Psychology of Mathematics Education (pp. 297-304). Mexico: PME.

Perret-Clermont, A.N. (1980). Social interaction and cognitive development in children. New York: Academic Press.

Piaget, J. (1970). Genetic epistemology. New York: Columbia University Press.

Saxe, G., Guberman, S., & Gearhart, M. (1988). Social processes in early number development. Monographs of the Society for Research in Child Development, 52 (20).

Skemp, R. (1976). Relational understanding and instrumental understanding. Mathematics Teacher, 77, 20-26.

Steffe, L.P., von Glasersfeld, E., Richards, J., & Cobb, P. (1983). Children's counting types:

Philosophy, theory, and application. New York: Praeger Scientific

Steffe, L.P., Cobb, P., & von Glasersfeld, E. (1988) Construction of arithmetical meanings and strategies. New York: Springer-Verlag.

Voigt, J. (1985). Patterns and routines in classroom interaction Recherches en Didactique des Mathematiques, 6, 69-118

THE ROLE OF PEER QUESTIONING DURING CLASS DISCUSSION IN SECOND GRADE MATHEMATICS

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Episodes of class discussion are used to illustrate the role played by peer questioning in constituting situations for children to learn what is acceptable as an explanation and to reflect on the nature of argumentation.

Previously we have elaborated the learning opportunities that arise and the nature of mathematical argumentation that occurs when children work collaboratively to solve instructional activities in mathematics (Krummheuer & Yackel, 1990; Yackel, Cobb, & Wood, in press). In this paper we focus on how peer questioning during class discussion provides opportunities for children's development of mathematical explanations and argumentation.

Classrooms which follow the Purdue Program for elementary school mathematics have as a goal to establish what we, following Richards (in press), have called the inquiry mathematics tradition as opposed to the school mathematics tradition (Cobb, Wood, & Yackel, in press). The inquiry mathematics tradition is characterized by children's active involvement in mathematical activity that fosters conceptual development rather than activity that encourages proficiency with rules and procedures that are demonstrated by the teacher or the textbook. A central aspect of classrooms in which we work is the negotiation of social norms which

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include that children figure out ways of solving problems that are personally meaningful, that they explain their solution procedures to others, that they listen to and try to make sense of the solution methods developed by their peers, and that they agree or disagree with solution methods proposed by other children (Cobb, Yackel, Wood, Wheatley, & Merkel, 1988). In some classrooms teachers have been successful in negotiating social norms that extend beyond agreement or disagreement to include that children question and challenge the explanations given by their peers during class discussions. Here we discuss the role of peer questioning during class discussion in children's developing ability to give mathematical explanations and arguments.

The teacher plays a crucial role in class discussions in that he or she "is the only member of the classroom community who can assess which of the students' constructions constitute a productive basis for further learning. ...the teacher ideally provides a running commentary on the students' constructive activities from his or her vantage point as an acculturated member of the wider community, but in terms that he or she infers are comprehensible to the students given their current mathematical ways of knowing." (Cobb, Yackel, & Wood, in press). By legitimizing certain solution methods but not others, the teacher helps children learn what is and what is not acceptable in the classroom, both in terms of what it means to engage in mathematical activity and in terms of what constitutes mathematical ways of knowing.

When classroom social norms include that children question each other's explanations during class discussion, the children play a crucial role as well in the process of developing mathematical explanations and arguments. Their questions function as challenges

for further explanation, and hence, the process of explaining and peer questioning is central to the interactive constitution of what is acceptable as an explanation in this setting.

Examples from the classroom

Typically we think of the teacher as serving the role of acculturating the children into the wider mathematical community. By encouraging the children to question each other, the teacher implicitly invites other children to participate with him in the acculturation process, by legitimizing certain solutions and sanctioning others. As they do so, the children have opportunities to focus on the nature of explanation. To question an explanation, it is necessary to reflect on the nature of the explanation and identify the aspect one wishes to question. The following example illustrates how children attempt to help each other understand what is accepted as an explanation. In this episode a child is attempting to explain his solution to 13 + 2 + 15 = ___. The problem had been posed in a whole class discussion format. Children were given time to solve the problem mentally, after which the teacher called on several of them to explain how they figured it out. The episode begins as Tony attempts to explain how he arrived at his answer of 30.

TY: I said 10 plus 2 equals 12.

T: 12

TY: And, and I said this 3 and this 2 equals

T: You used the 2 once now, didn't you.

TY: Yes, and and 5 plus 3 equals 8.

T: Eight. Okay.

TY: And I said this one

T: Is that really a one?

TY: No. It's 10.

T: Okay

TY: And, and 10 plus 50, wait a minute, uh, and I said this, this, um, wait, this, this 3 is 30 and

T: You have some questions.

In the next portion of the episode Tony calls on several children who have their hands up. At this point the teacher withdraws from interacting with Tony about his mathematical activity to allow the other children to assume this role. The teacher maintains the conversation by reminding Tony to call on children who have questions.

DR: Uh, where did you get the 30 from?

TY: From the 3, and, and I said ---

T: Tony, look, you have some questions

TY: Charles?

CH: Where did you get the 50 from? You said 50 and ...

TY: Uh, from the 30 and, and I got 30. (pause)

T: Tony, call on someone, they're waiting on you. They have questions.

TY: Um. Mark.

MK: Don't that 30 supposed to be a 3, though?

TY: Um, yes.

MK: Then why you calling it 30?

...

CH: That's supposed to be 3 ones, you keep calling it thirty.

The children are attempting to help Tony understand that his explanation doesn't make any sense to them. His attempt to solve the problem by operating with the digits is confused by his lack of

understanding of place value and his inability to meaningfully decompose a number. His interpretation of the digits in the problem differs from the conventional interpretation of place value notation, which is taken-as-shared by a majority of the children in the classroom. The other children are reinforcing the notion that explanations in this classroom must be consistent with taken-as-shared interpretations.

By participating in the process of questioning the children also learn to evaluate the explanations of others. This extends beyond evaluating the solution methods described to evaluating the reasoning involved in the questions and challenges posed. To do so requires that children distance themselves from the problem solving activity and focus their attention on the form of the argumentation used. The following episode, taken from a class discussion in April, illustrates the difference between focusing on the solution process and focusing on the form of the argument. The teacher has posed the question 46 + 38 + 54 = ___ in a whole class format. The episode begins as Donelle attempts to explain how she arrived at her answer of 138.

DL: I said 40 plus 30. That'd be 70.

T: Okay.

DL: And I said 70 plus the 8, that'd be 78.

T: Okay.

DL: And I said 78 plus the 50, that'd be (pause) a hundred and --- sixteen.

Several students put up their hands. Donelle calls on Travonda who ask her if she would add it up again. The issue is the sum of 78 and 50. Donelle perform the computation again mentally and again gets 116 as the answer.

TV: ...you said that was 16, what other-- when you add 6 and the 4 and you won't -- you'll get 26 and you, you won't get, um 138.

DL: (to the teacher) I didn't understand her question.

T: (to TV) What's the question?

TV: I, I said if you said that was 116, and I said the 6, that'd be 116 and 6 plus 4, that'll equal to 10 so the 116 plus the 10, that will make it, um, a hundred, um, a hundred and twenty-six, and that won't be, um, 138.

While we would anticipate that some child might challenge the result of Donelle's computation, the nature of Travonda's argument is remarkable since it takes the form of a proof by contradiction. Although Donelle does not know how to interpret it, Travonda's argument is potentially productive since it provides other children with an opportunity to reflect on the form of her argument. Jameel poses a question to Travonda. He asks where the 126 in her argument came from. Now the subject of the conversation has changed from Donelle's explanation to Travonda's argument.

JL: Wouldn't that be a, that would be a 128? ... I'm, talking about when TV said it was 126.

TV: Yea, but I'm talking about when she said 116 and I said she only had the 6 and the 4 left, so that would make it a 10 and with that 10, and she add it to the 116 she'll get 126 and she wants to get, um, 138.

The discussion continues as two other children explain their methods of solving the original addition problem. Subsequently, Jameel returns to his discussion with Travonda.

JL: (to the teacher) I don't understand her [TV's] question.

T: Well what question?

JL: What she [TV] asked.

Jameel clearly understands that the obligation to give an explanation that is meaningful to the class applies to Travonda as well as to Donelle. As the episode continues the teacher is ready to dismiss Jameel's comment but Travonda ignores the teacher and accepts the obligation to explain once more.

T: Well that's, that's gone now. We've forgotten it.

TV: He's [JL is] talking about when I said that it would be 126. If she would've added the 6 and the 4, that'll be 126 and, and the last number, um and she said it would be 138, so she would, so she, and that'd be 126 so she wouldn't get 138.

JL: Well, what about the 10?

TV: I added it. She said it'd be 116 and I said--that'd be a hundred and sixteen. That'd make 126.

We contend that the opportunity for Jameel to focus on the form of argumentation Travonda used was directly related to the fact that he, unlike Donelle, was not personally engaged in the process of explaining his solution to the addition problem. He was able to distance himself from the process of solving the problem and focus on the nature of the challenge posed.

In these examples we have illustrated that through peer questioning children are involved in determining what counts as an acceptable explanation in the classroom and that they develop sophisticated forms of explanation and argumentation which are influenced by the social interaction of the participants. These arguments provide opportunities for other children to reflect on the nature of the argument as well as on the solutions to the instructional activities. The role played by peer questioning is that is sets the stage for the interactive constitution of

situations for explanation and argumentation that are not present when children work alone or in pairs.

References

Cobb, P., Wood, T., & Yackel, E. (in press). Characteristics of classroom mathematics traditions: An interactional analysis. In C. Maher & R. Davis (Eds.), Relating schools to reality in mathematics learning. Engelwood Cliffs, NJ: Prentice Hall.

Cobb, P., Wood, T., & Yackel, E. (in press). Interaction and learning in mathematics classroom situations. Educational Studies in Mathematics.

Cobb, P., Yackel, E., Wood, T., Wheatley, G., & Merkel, G. (1988). Research into practice: Creating a problem solving atmosphere. Arithmetic Teacher 36(1), 46-48.

Krummheuer, G., & Yackel, E. (1990). The emergence of argumentation in the small group interaction of second graders. In G. Booker, P. Cobb, & T. deMendicuti (Eds.), Proceedings of the Fourteenth Conference of the International Group for the Psychology of Mathematics Education, Vol III, (pp.113-120). Mexico: PME.

Richards, J. (in press). Mathematical discussions. In E. von Glasersfeld (Ed.), Constructivism in mathematics education. Holland: Reidel.

Yackel, E., Cobb, P., & Wood, T. (in press). Small group interactions as a source of learning opportunities in second grade mathematics. Journal for Research in Mathematics Education.

The effect of graphic representation: An experiment involving algebraic transformations

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Summary

This study examined the effect of graphic representation of algebraic expressions on performance of tasks involving transformations. Student teachers, who received instruction using multiple representation software, were tested on their ability to debug faulty transformed expressions, either with or without graphic feedback. Two main strategies were identified while debugging - Syntactic Manipulation (SYM) and Semantic Interpretation (SEI). It was found that subjects who used the SEI strategy, significantly improved their performance while using graphs, as compared to no use of graphs. Some positive effect of the graphic feedback was also manifested for those using the SYM strategy, who were assisted by the graphs but continued to syntactically manipulate the expressions. It was concluded that effective use of the information provided by the graphic representation, accompanied with better understanding of the deep structure of expressions can induce better performance on various algebraic tasks.

In a recent study (Yerushalmy 1991) I investigated the effects of various computerized environments on students' ability to carry out algebraic transformations and to debug their own manipulating processes. The methods used by beginner algebra students to correct their own mistakes were observed, as was their use of and reaction to immediate computerized feedback, offered by the Transformer software (1991, 1989), which indicated they had made an error. One of those feedbacks was the difference graph which provided the graph of the difference (if any) between original and transformed expression. In general I found four ways of relating to the difference graph feedback: Students either ignored the graph and the information provided by it; used the graph as an icon without identifying the appropriate information (e.g. if the difference graph was not the f(x)=0 graph, they knew they had made a mistake, but did not try to get more information about the type of the difference graph (quadratic, linear etc)); used the graphic feedback for control and analysis, but did not actually implement it in the algebraic transformations; or related to the semantic structure of the expression by analyzing and using the graphic feedback while transforming (e.g. they were able to identify that when the difference graph is a linear function, they must have made a mistake in terms and operations that could result in a linear or a constant term).

The present study attempts to complement and expand the information provided by the former research. Its aim is to reveal effects of provision of functions' visual representation, on the performance

of students who are well familiarized with both the symbolic and graphic representations of expressions, on tasks involving algebraic transformations. Specifically, the study examines:

1) The strategies used by students while debugging faulty transformed expressions using computerized graphic feedback.

2) The relation between the strategy used and various parameters measuring the quality of performance (e.g. duration of time, number of errors) on similar tasks performed with or without graphic feedback.

3) The effect of transforming expressions with the use of graphic representation on the understanding of functions and expression measured by traditional paper and pencil tasks.

Method

Twenty eight elementary school math teachers enrolled to an algebra course, participated in the study. The experiment consisted of three main phases:

Phase I: **Teaching functions using multiple representation**

This phase lasted 9 sessions in which students learned, using the Function Analyzer (1989), about functions' properties and operations with functions using the Function Supposer (1990). During all sessions, special emphasis was made on the connection between the two modes of representation - the graphic and symbolic. Subjects worked on projects in pairs, followed by discussions in a whole group setting.

Phase II: **Teaching transformation of algebraic expressions using single and multiple representation**

Students used the Transformer software for five sessions in which they worked in pairs on a file of 10 problems ordered by increasing complexity. On each problem they had to transform a given expression into a target form of the same expression (e.g. 2(x+3(x+1)) had to be transformed into Ax^2+Bx+C). Each student used the graphic and the error message feedbacks alternately between sessions.

Phase III: **Testing and interviewing**

1) Paper-and-pencil pre/post tests:

A written test was administered before and after phase II of the experiment, which consisted of two parts: 1) function identification (FI), which was related to the topics learned on the first phase. 2) algebraic manipulation (AM), which included four expressions to be transformed and was related to the second learning phase.

2) Computer test

Following the posttest, each participant took a test with the TRANSFORMER , which consisted of a file of eight items of the following structure: A given expression was faulty transformed into another expression and both the given and mistaken expressions were presented (e.g. the expression a*(b+c)*d was faulty transformed into a*b+a*c*d). The student's task was to indicate the mistake, write the correct transformed answer and then to describe in pencil and paper the method used.

The file included four types of erroneous processes, each appearing once in the error message TRANSFORMER (labeled as NG items) and second in the GRAPH version (labeled as G items).

:rrors chosen were identified in the literature as well as in previous experiments with the TRANSFORMER (Yerushalmy 1991) as major sources of difficulties and indicates of fragile knowledge. While students were working with each version of the TRANSFORMER, records were kept about each transformed expression and the time spent on each step for each item. These records served as the main source for analysis of the research results.

3) Interviews

Two weeks after the computer test 11 students were individually interviewed. The students chosen were mainly those of whom the computer records were not consistent with their self- reports during the test. During the interview subjects solved two items taken from the computer test - one with any feedback they chose, and the other with the graphic feedback.

Results

Categorization of strategies and groups

The records documenting students' dribble files while solving the G items of the computer test (i.e. with the graphic feedback), were used for determining two main strategies: while trying to identify and correct the deliberately mistaken expression (which was the test's requirement) students could either:

1) Restart over from the given expression and transform it to the simplest polynomial; then identify and correct the given mistake by comparing it to their final result. For example:

GIVEN: -x(7 - 2x) - 2 - 3x(5 - 2x(3 - x))

MISTAKEN EXPRESSION: 6x^3 + 20x^2 - 22x - 2 (instead of: -6x^3 + 20x^2 - 22x - 2)

STUDENT'S RECORDS:

-7x + 2x^2 - 2 - 3x(5 - 6x + 2x^2)

-7x + 2x^2 - 2 - 15x + 18x^2 - 6x^3

-6x^3 + 20x^2 - 22x - 2

This strategy, operating technically on the syntax of the expression will be called "syntactic manipulation" or in short SYM.

2) Correct the given mistake by changing mainly the relevant terms that could have caused the error, without direct
manipulation of the given expression. For example:

GIVEN: -x(7 - 2x) - 2 - 3x(5 - 2x(3 - x))

MISTAKEN EXPRESSION: 6x^3 + 20x^2 - 22x - 2 (instead of: -6x^3 + 20x^2 - 22x - 2)

STUDENT'S RECORDS:

-6x^3 + 20x - 22x - 2

-6x^3 + 20x^2 - 22x - 2

This strategy, relating to the deep structure or semantic aspect of the expression will be called "semantic interpretation" or in short SEI.

These strategies are in accord with findings of researches involving debugging in computer programing (Katz & Anderson 1987-88), and were also observed in my previous study mentioned above (Yerushalmy,

1991).

The response to each item on the Graph part of the computer test was assigned one of the two strategies according to the above descriptions. On this basis, students were split into three strategy groups:

1) SYM group - composed of students who used the "syntactic manipulation" strategy on all four items (11 students).

2) SEI group - composed of students who used the "semantic interpretation" strategy on all four items (11 students).

3) MIS (mixed strategy) group - composed of students who used both strategies inconsistently (6 students).

These strategy groups were highly correlated with ability level the SYM group and the SEI group consisting mainly of low and high ability students respectively.

I. Performance on the computer test

Four parameters were measured in order to evaluate the performance of each strategy group on the NG and G items: the number of steps and time needed to solve each item, the number of self errors made and the time needed to correct such an error. Results of these four parameters are as follows:

Strategy Group	SYM		MIS		SEI	
	NG	G	NG	G	NG	G
Average # of steps	3.9	3.4	3.2	2.4	2.22	1.4
Average time duration (sec)	533	363	387	320	412	210
Average number of errors	8.45	4.63	1.50	2.66	4.3	1.54
Average time duration to correct errors (sec)	137	81	80	143	136	84

In sum, the parameters comparing the performance of both the SEI and SYM strategy groups on the G part of the computer test to its NG part reveal an advantage to the use of graphic representation in identifying and correcting transformation mistakes (since the differences on all parameters were non-

significant for the MIS group it is not referred to here). This benefit from the graphs was most profound for the SEI students, who seemed to be using them effectively. However, it was also notable for the SYM group, which exhibited improvement on all parameters except for the number of steps. This is an unexpected result, since they seemed to be only manipulating the expressions technically, without referring to the graphs. The protocols of the interviews shed some light on this unanticipated finding.

II. Findings from the interviews

The students interviewed were mainly from the SYM group who reported on using graphs in their solution. Their interviews show that they actually did use graphs to solve the problems, but for various reasons, such as lack of confidence, or conservatism, preferred to use the conventional method of manipulating expressions. This finding could explain the benefit from the graphic representation demonstrated for the SYM group. Interviews with students of the MIS group revealed they were willing to spend long time of trial and error to work with both symbolic and graphic representations, even at the cost of making errors, which might explain their relatively poor results on the computer test. Another finding from the interviews is that all students were able to operate with the graphs and use them effectively in solving the problems, when required to do so.

III. Results of pre and post tests

Following are the percentages of correct answers to the paper and-pencil tests (which include function interpretation items (FI) and algebraic manipulation items (AM) administered before and after the experimenting phase:

Strategy Group	SYM		MIS		SEI	
	Pre	Post	Pre	Post	Pre	Post
FI	74	91	87	91	83	95
AM	35	83	52	91	53	91

Overall findings indicate better performance of all groups on the posttest compared to the pretest, on both parts of the test. For the AM part of the test this is not a surprising result, since students were engaged in algebraic manipulations through the whole experimental phase. For the FI part, however, the positive effect is not trivial, since the topic of functions' properties was not directly taught.

Discussion

The results of this study indicate a notable positive effect of a graphic feedback as compared to a verbal error-message feedback on students' performance associated with transforming expressions. The most profound effect of the graphic representation was found for the SEI group - those students who used semantic interpretation strategy in order to debug faulty transformed expressions. Better performance on the G compared to the NG items was also evidenced for the SYM group. These students, who seemed to be using only the syntactic manipulation strategy, actually understood and considered the graphs, but refrained from using their information directly while transforming. Instead, they preferred to continue with the traditional more conservative method.

The use of linked symbolic and graphic representation environment was also found to improve all students' achievements on traditional paper and pencil tasks. This finding, taken with the evidence of effective use of graphs on the computer test items which were not encountered by students before, show that knowledge and understanding of both the graphic and symbolic aspects of functions acquired in the learning phase, were transferred and implemented in various tasks. Thus, operating on algebraic expressions with the use of computerized multiple representation environment, allows better performance by understanding them as functions and realizing their deep rather than surface structure. The present study was done with experienced teachers which were probably influenced and maybe even "trapped" by their former knowledge and methods they learned prior to the experiment. Thus, an interesting question for further research is whether and how can beginning algebra students (as in the first study 1991) be taught by integrating functions and expressions.

References

Katz, I.R. and Anderson, J.R. (1987-88) Debugging: An analysis of bug-location strategies. *Human Computer Interaction*, vol. 3, pp. 351-400

Schwartz, J.L and Yerushalmy, M. (1989) *Visualizing Algebra: The Function Analyzer* (computer software), Education Development Center and Sunburst Communications, Pleasantville, NY.

Schwartz, J.L and Yerushalmy, M. (1990) *Visualizing Algebra: The Function Supposer* (computer software), Education Development Center and Sunburst Communications, Pleasantville, NY.

Yerushalmy, M. (1991) Effects of computerized feedback on performing and debugging algebraic transformations, *Journal of Educational Computing Research*, vol. 7 (to appear)

Yerushalmy, M. (1989) The use of graphs as visual interactive feedback while carrying out algebraic transformations, *Proceeding to the 13th international conference of the Psychology of Mathematics Education*, pp. 252-260, Paris.

IN WHAT WAYS ARE SIMILAR FIGURES SIMILAR?

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Similarity is considered one of the more difficult topics to teach and to learn in the geometry curriculum in secondary school. The concept of similarity is often addressed in a more intuitive way much earlier - in the elementary school curriculum. A trace is presented that moves from an analysis of a small portion of subject-matter knowledge possessed by an experienced fifth grade mathematics teacher with respect to similarity, to an analysis of how that knowledge is reflected in the actual presentations in class, and then to the effects of those presentations on four of his students. There seem to be links between students' misconceptions and their teacher's misconceptions. These misconceptions might impede the study of similarity at a more advanced level in higher grades.

Introduction

Teachers who possess an incomplete knowledge base might provide their students with incomplete descriptions of concepts and interconnections within a specific domain. This, in turn, may lead to an incomplete knowledge base of the student. To trace this influence we would need to closely trace the understanding a teacher has of a concept, through the way s/he presents it in classroom, and then finally to the understanding of this particular concept that the student develops throughout instruction.

The impact of instruction, in terms of students' knowledge and understanding, is usually assessed at the end of an instructional unit and/or at the end of the school year. Furthermore, assessment of student knowledge often addresses skills students are expected to perform rather than conceptions they hold. Given the lag between instruction and assessment, it is practically impossible to trace results of such assessments, whether or not they indicate competence of the student, back to particular instructional moves or to teachers' knowledge that may have influenced those moves. To be able to come up with such a trace, students' specific content knowledge needs to be closely

examined at short intervals of time throughout instruction, similarly to Leinhardt's (1988) work. In this paper such a trace is presented. The underlying concept for this analysis is geometric similarity.

The Study

The data used for this study is part of a larger set of data that was collected in order to describe and analyze the teaching and learning of graphing and coordinate geometry in elementary school (Stein & Baxter, 1989; Zaslavsky, 1989).

Subjects

An experienced mathematics teacher (17 years of teaching experience) was videotaped as he taught a unit on graphs and coordinate geometry. The notion of similar figures was introduced as part of coordinate geometry. The teacher had been teaching this unit for over five years. The teacher was highly recommended by his building principal and fellow teachers for his reputation as a good mathematics teacher.

Four students participated in the study. Two of them, Carl and Carmen, were classified by the teacher as top level students. Carl participated in the district's gifted program. The other two, Cindy and John, were classified by him as below average students. The teacher recommended these students as qualifying for the requests of the researchers. He was asked to take in consideration not only students' competence, but also their degree of articulation.

Data collection

Data was collected from four sources: 1) Videotapes of actual classroom instruction; 2) A teacher stimulated recall interview; 3) Pre and post unit interviews with students; 4) After-class interviews with students.

Results

The results will be discussed in terms of the teacher's knowledge about similarity and contrasted with what is commonly considered within the mathematics discipline as a more complete and correct knowledge. Next, some instructional moves of the teacher, that were identified as particularly reflective of his content knowledge, are described and connections are made indicating possible knowledge components that they reflect. Finally, results of students' responses to interview questions are reported and speculations are made

regarding possible links between students' understandings of the content taught in class and the instruction that took place.

Teacher's subject matter knowledge

Assessing teachers' knowledge in general, and experienced teachers in particular, is a very delicate task, given the uneasiness and tension that teachers express when presented with questions that aim to test their knowledge. Therefore, inferences about the knowledge teachers have must often be drawn indirectly from discussions with them and from their instructional actions (Leinhardt, 1987). In this study the analysis of the teacher's subject-matter knowledge was based on a stimulated recall interview, which was designed and structured in a way that encouraged him to reflect on and talk about his conceptions of similarity and similar figures.

The teacher used a very intuitive definition to describe similar figures, which is an incomplete version of the textbook definition. In order to decide whether two figures are similar he first establishes that they are the same shape and then makes sure that they are not the same size. His working definition relates to the following definition which often appears in elementary level textbooks: *Two figures are similar if they have the same shape but* ***not*** ***necessarily*** *the same size.* The teacher ignores the qualifier *"not necessarily"*. He does not include congruent figures as a special case of similar figures, because congruent figures have, in addition to the same shape, also the same size.

The teacher does not seem to know that a crucial part of the definition of similarity relates to ratio and proportion. Thus, to him all rectangles are similar, provided they are not congruent.

Since *"the same shape"* is not a clearly defined mathematical notion, the teacher decides in a very intuitive and inconsistent way whether or not two figures have the same shape. Thus, according to the teacher, a rectangle and a square do not have the same shape, even though they have some similar features. Furthermore, to him all trapezoids have the same shape, thus, any two are either congruent or similar. The teacher refers mainly to similar polygons, yet does not once relate similarity to angles of polygons. It seems that he is not familiar with the *necessary condition* regarding congruent angles, i.e., that there must be a correspondence between vertexes of two similar polygons under which any pair of corresponding angles are congruent (this is not a sufficient condition for any two polygons, unless they are triangles). The teacher classifies triangles into three types, each type having *"the same shape"*

Thus, to him all equilateral triangles are similar (as follows from the mathematical definition of similarity), all isosceles triangles are similar, and all right angle triangles are similar.

To summarize, the teacher's subject-matter knowledge of similarity is basically vague, incomplete and erroneous.

Classroom Instruction

The analysis of classroom instruction is based on transcripts of actual classroom instruction (two lessons dealt with the notion of similarity), as well as notes taken in class. The focus of this analysis is on the specific examples (Rissland, 1978) and tasks the teacher chose to present to his students, and on the explanations he provided for justifying answers to questions or solutions to problems. It has been established that this particular teacher heavily relies on the textbook (Stein & Baxter, 1989), which has bearing on his instructional moves.

After introducing the full textbook definition of two similar figures (see above), the teacher presented a number of examples illustrating the definition. He started with two similar objects: two balls, a tennis ball and a basketball. The second example the teacher presented was a pair of rectangles which were placed on a bulletin board in class. He used them to show that all rectangles are similar, even though these particular rectangles were not similar, because their corresponding sides were clearly not proportional. Then the teacher moved on to an example suggested in the textbook - a photograph and its enlargement. However, in the explanation he gave the teacher referred to the frame of the picture being rectangular, thus having the same shape and different size, and did not mention the relation between the two pictures. This could be seen as a missed opportunity to discuss, or at least to mention, the underlying proportionality of similar figures. Then the teacher presented two congruent hexagons which he had cut out earlier, and by showing that they can overlap, he made sure the students did not classify them as similar figures, since they had the same size. It is quite obvious that the teacher tried to highlight the difference between congruence and similarity, even though it was inconsistent with the definition that he cited at the beginning, which included congruence as a special case of similarity.

After these four initial examples, Instruction focused on two types of tasks: One type of task was a construction task, in which students were supposed to draw in a coordinate grid a figure similar to another given figure. These tasks were textbook tasks. The procedure for constructing a similar figure

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required changing the scale of the axes, which again had to do with the fact that corresponding sides of similar polygons are proportional. The students were told how to carry out the procedure without any explanation why it is done in such a way - perhaps another missed opportunity. The second type of task was a classification task, in which students were supposed to select one figure (out of three), which is similar to a given figure. These tasks were taken as is from another workbook. Some of these tasks were misleading (e.g., in several cases none of the figures were similar to the given one, while by method of elimination, the teacher drew wrong conclusions, which his students followed). For a teacher who heavily relies on textbooks and does not question their correctness, and at the same time has a very limited subject-matter knowledge, such a worksheet could be quite harmful.

To summarize, the teacher's subject-matter knowledge seems to be reflected in the classroom instruction in different ways. The examples he presented seem to reflect his knowledge about criteria for deciding whether two figures are similar, some of which are not valid. His limited knowledge prevented him from making connections and providing appropriate explanations. In the next section links between classroom instruction and students understandings are suggested.

Students understandings

Students' understandings of the notion of similar figures were analyzed based on the pre and post unit interviews, as well as on the after class interviews. All four students seemed to have mastered the procedure they were taught in order to draw a similar figure to a given figure in a coordinate grid. None were able to explain why the procedure worked. As to classification tasks, Table 1 presents students' answers to one of the classification tasks presented in the post unit interview (in the interview all figures were drawn on a grid paper, so the students were able to easily measure lengths, if they wished to). It is quite obvious that all four students did not accept congruent figures as similar ones (see first row). It is also clear that all students were able to identify correctly similar figures (second row). In other cases, i.e., for none similar figures, each student built his or her own conception of what *"the same shape"* meant. Particularly interesting is the resemblence between the two girls' answers (Carmen and Cindy) and the difference between their answers and the boys' answers (Carl and John). The girls seem to have developed a more adeqaute concept, though it is less related to classroom instruction (perhaps they learned in spite of the teacher). The girls seemed to have induced that angles play a role in *"shape"* (they talked about "steepness"), while the boys

did not treat angles as a factor that influences *"shape"* (see rows 3 to 5). Carl, who was the top student in his class, and a very capable one, interpreted the notion of *shape* in a very general and abstract way: To him the shape of a figure does not change if instead of connecting its vertexes by straight line segments they are connected by curved lines (see row 6).

To summarize, the teacher's incomplete knowledge of similarity was reflected to a different extent in all four students' answers and explanations. Each one developed his or her own interpretation of what the fuzzy notion of *"shape"* meant. It can be speculated that when these students reach the stage in which they will be required to learn the topic of similarity in their geometry course in secondary school, some of them will have to go through an "unlearning" process.

Discussion

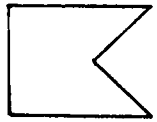
Given that similarity is a difficult topic to learn at secondary school (Chazan, 1987; Kumpel, 1975), it could be argued that it is important to try and build students' intuitive understanding of the notion of similarity at earlier stages of the curriculum. However, if teaching similarity is a non trivial task, how can elementary school teachers be expected to facilitate the development of students' intuitive conceptions of similarity in a way that secondary school teachers could build on?

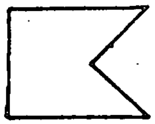
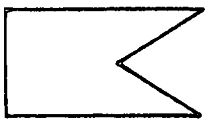
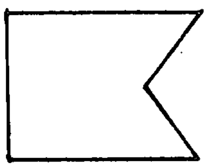
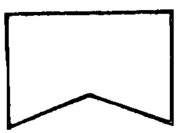
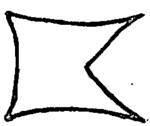
Given that a teacher's misconceptions and limited subject-matter knowledge are likely to be reflected in some of his or her students, how can we ensure that a teacher possesses the knowledge necessary for teaching mathematics without limiting or impeding students' understanding?

Table 1: Students' responses to a classification task on the post interview

(+) designates a correct answer; (-) designates an incorrect answer.

The task - The following figure is given:



Is this figure similar to the given figure?	Carl above average	Carmen above average	Cindy below average	John below average
 congruent	no (-)	no (-)	no (-)	no (-)
 similar	yes (+)	yes (+)	yes (+)	yes (+)
 not similar	yes (-)	no (+)	no (+)	yes (-)
 not similar	yes (-)	no (+)	no (+)	yes (-)
 not similar	yes (-)	no (+)	no (+)	yes (-)
 not similar	yes (-)	no (+)	no (+)	no (+)

References

Chazan, D. (1987). Similarity: Unraveling a conceptual knot with the aid of technology. In In J. S. Bergeron, N. Hercovics, & C. Kieran (Eds.), Proceedings of the eleventh international conference of the International Group for Psychology of Mathematics Education (pp. 3-9). Montreal, Canada.

Friedlander, A., Lappan, G. (1987). Similarity: Investigations at the middle grades level. In M. M. Lindquist & A. P. Shutle (Eds.), Learning and Teaching Geometry, K-12. (pp. 136-143). The National Council of Teachers of Mathematics, Inc. Reston, Virginia.

Kumpel, P. (1975). Do similar figures always have the same shape? Mathematics Teacher, 68(8), 626-628.

Leinhardt, G., (1988). Getting to know: Tracing students' mathematical knowledge from intuition to competence. Educational Psychologist, 23(2), 119-144.

Leinhardt, G. (1987), April). What does subject matter knowledge buy? Paper presented at the annual meeting of the American Edaucational Research Association, Washington, DC.

Rissland, E. L. (1978). Understanding understanding mathematics. Cognitive Science, 2(4), 361-383.

Stein, M.K., & Baxter, J. (1989, March). Teachers use of texts. Paper presented at the annual meeting of the American Educational Research Association, San Francisco.

Zaslavsky, O. (1989, March). Development of a mathematical concept: A trace from teacher to student. Paper presented at the annual meeting of the American Educational Research Association, San Francisco.

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(in: L. Villani, Leonardo a Milano, Musumeci Ed.)

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